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DIFFERENTIAL-GAME EXAMINATION OF OPTIMAL TIME-SEQUENTIAL FIRE-SUPPORT STRATEGIES

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Monterey, California

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by

James G. Taylor

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Optimal time-sequential fire-support strategies are studied through a two-person zero-sum deterministic differential game with close-loop (or feedback) strategies. Lanchester-type equations of warfare are used in this work. In addition to the max-min principle, the theory of singular extremals is

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) required to solve this prescribed-duration combat problem. The combat is between two heterogeneous forces, each composed of infantry and a supporting weapon system (artillery). In contrast to previous work reported in the literature, the attrition structure of the problem at hand leads to force-level-dependent optimal fire-support strategies with the attacker's optimal fire-support strategy requiring him to sometimes split his artillery fire between enemy infantry and artillery (counterbattery fire). A solution phenonomenon not previously encountered in Lanchester-type differential games is that the adjoint variables may be discontinuous across a manifold of discontinuity for both players' strategies. This makes the synthesis of optimal strategies particularly difficult. Numerical examples are given.

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1. Introduction.

The allocation of a specific weapon system type to an acquire target is an important tactical decision in the fire-support process.† Accordingly, the determination of optimal (or even good) fire-distribution strategies for supporting weapon systems†† is a major problem of military operations research. The problem is of interest to the military tactician so that he may have a clearer understanding of the circumstances under which a supporting weapon system (such as artillery) should engage the enemy's primary weapon system (i.e. infantry) and when it should engage the enemy's supporting weapon systems.

In this paper we will examine the dependence of optimal time-sequential fire-support strategies on the form of the combat attrition model. Previous work by Weiss [38] and Kawara [22] suggests that an optimal fire-support strategy consists in always concentrating all fire on one enemy target type (although this target type may change over time). We will consider a differential game with slightly different combat dynamics than the fire-support differential game recently considered by Kawara [22] and show that optimal fire-support strategies quite different in structure than those obtained by him may arise. Moreover, the solution to the problem which we consider in this paper involves a solution phenomenon not previously encountered in a Lanchester-type differential game:††† the dual (or adjoint) variables may be discontinuous across a manifold of discontinuity for both players' strategies.

Fire-support operations (as are any combat operations) are a complex random process (see [26]). We will nevertheless consider a simplified deterministic

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†See pp. I-33 to I-43 of [26] for a further discussion.

††See [38] for a brief discussion of the distinction between a "primary" weapon system (or infantry) and a "supporting" weapon system.

†††We refer to a differential game as being a Lanchester-type differential game when the system dynamics are described by Lanchester-type equations of warfare (see [34]).

Lanchester-type model in order to develop insights into the structure of optimal time-sequential fire-support strategies. H. K. Weiss [38] has emphasized that such a model of an idealized combat situation is particularly valuable when it leads to a clearer understanding of significant relationships which would tend to be obscured in a more complex model.

The problem of determining an appropriate mixture of tactical and strategic forces (another aspect of the optimal fire-support strategy problem) was extensively debated by experts during World War II. Some analysis details may be found in the classic book by Morse and Kimball (see pp. 73-77 of [27]). The problem was studied at RAND in the late 1940's and early 1950's (see [16]) and elsewhere (see [1]). It would probably not be too far-fetched to claim that this problem stimulated early research on both dynamic programming (see [2]) and also differential games (see [16], [20]). Today the problem of determining optimal air-war strategies is being extensively studied by a number of organizations (see, for example, [17], [25], [29], [36]). An idealized version of A. Mengel's problem (see [16]) appears in Isaacs' book as the "War of Attrition and Attack" (see pp. 96-105 of [21]). Discrete-time versions of this problem of determining optimal "air-war" strategies have been considered by a number of workers as time-sequential combat games [5], [6], [15] (see also [7], [13]). A related problem has been considered by Weiss [38] (see also [37]), who studied the optimal selection of targets for engagement by a supporting weapon system.† More recently, Kawara [22] has studied optimal time-sequential strategies for supporting weapon systems in an attack scenario version of Weiss' problem. Other recent work has considered various conceptual and computational aspects of time-sequential combat games [28], [29], [30].

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†See [33], however, for a justification of the optimality of strategies given by Weiss [38]. A general solution algorithm is also presented in this paper [33].

Since our work here may be considered to be an elaboration upon and extension of Kawara's fire-support differential game [22], we will review his main results and relate our work here to them. Kawara [22] considers combat between two heterogeneous forces, each composed of infantry (the primary weapon system) and artillery (the supporting weapon system). The time-sequential decision problem is to determine each side's optimal strategy for distributing its supporting weapon system's fire over enemy target types according to the criterion of the infantry force ratio at the prescribed-duration attack's end. Kawara concludes that each side's optimal† strategy is to always concentrate all supporting fire on the enemy's supporting weapon system (counter-battery fire) during the early stages of battle (if the total prescribed length of battle is long enough) and then later to switch to concentration of all fire on the enemy's infantry. He states that this switching time "does not depend on the current strength of either side but only on the effectivenesses of both sides' supporting units" (see p. 951 of [22]). Moreover, an optimal strategy has the property of always requiring concentration of all supporting fire on enemy infantry during the final stages of battle.

Thus, Kawara concludes that for his model the optimal fire-support strategies do not depend on force levels. However, this is only true provided that the appropriate side's (in Kawara's numerical example, the defender) supporting weapon system is not reduced to a zero force level before a critical time.†† Let us assume, therefore, that neither side's supporting weapon system

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†Kawara does not determine the optimality of extremal strategies determined for his problem (i.e. show that sufficient conditions of optimality are satisfied (see [4])). We use the work extremal to denote a trajectory on which the necessary conditions are satisfied.

††See the expression for T_2'' on p. 949 of [22] and its plot in Figure 4 of [22].

can be reduced to a zero force level.† For this condition the optimal fire-support strategies are force-level independent and may be expressed solely in terms of "time-to-go" in the prescribed duration battle. The purpose of this paper is to show that a tactically realistic variation in the attrition equations leads to a problem with force-level dependent optimal fire-support strategies. This result has an important implication for tactical decision making: optimal time-sequential allocation of fire-support resources depends on not only initial intelligence estimates but also on a continuous monitoring of the evolution of the course of combat.

Thus, the purpose of this paper is to illustrate the dependence of optimal fire-support strategies on the nature of Lanchester-type combat attrition equations (see [34]). We consider a slight variation in Kawara's problem (i.e. different combat dynamics) for which the structure of the optimal strategy of one of the combatants is significantly different than that in the original problem [22]: the optimal strategy of one combatant depends directly upon the enemy's force levels and is no longer to always concentrate all fire on either the enemy's primary or supporting weapon system. Furthermore, we will show that an optimal strategy in which a side divides the fire of its supporting weapon system between the enemy's primary (infantry) and supporting systems can only occur when the enemy's infantry has some fire effectiveness (in the sense of a non-zero Lanchester attrition-rate coefficient) against his infantry. The optimal strategy of one side to sometimes split its fire is very similar to that which occurs in a one-sided (optimal control) problem previously considered by us [31] (see also [32]) for the optimal distribution of fire by a homogeneous force in combat against homogeneous forces. In [31] the enemy consisted of two weapon system

†Initial force levels and the known length of battle may be sufficient to guarantee this for a given set (or range of values of) Lanchester attrition-rate coefficients.

types, each of which undergoes attrition at a rate proportional to the product of the numbers of firers and targets (referred to, for convenience, as "linear-law" attrition). In fact, this previous work of ours [31] was the motivation for our examination here of other attrition structures in Kawara's problem.

2. Kawara's Fire-Support Differential Game.

Since Kawara's fire-support differential game is the point of departure for this paper, we will review the development of his model. The reader will find it convenient to compare the mathematical statement of Kawara's problem (1) with the fire-support differential game studied in this paper (2) in order to understand the dependence of optimal fire-support strategies on the mathematical form of the attrition equations.

Kawara [22] considers the attack of heterogeneous X forces against the static defense of heterogeneous Y forces. Both the X and Y forces are composed of two types of units: primary units (or infantry) and fire support units (or artillery). The X infantry (denoted as X_1) launches an attack against the Y infantry (denoted as Y_1). We consider that phase of the attack which may be called the "approach to contact." This is the time from the initiation of the advance of the X_1 forces towards the Y_1 defensive position until the X_1 forces actually make contact with the enemy infantry. It is assumed that this time is fixed and known to both sides and that infantry fire has negligible effectiveness against the enemy's infantry during this time. During this time the fire support units remain stationary, and each unit has the capability to deliver either "point-fire" counterbattery fire against enemy artillery or "area fire" against the enemy's infantry.

It is the objective of each side to attain the most favorable infantry force ratio[†] possible at the end of the "approach to contact" at which time the

[†]See [35] for some insights into the dynamics of combat from considering the force ratio.

force separation between the opposing infantries is zero and artillery fires must be lif. ed from the enemy's infantry in order not to also kill friendly forces. Thus, the decision problem facing each commander is to determine the "best" distribution of artillery fire over time between enemy infantry and enemy artillery in order to maximize the quotient of friendly infantry (numerical) strength divided by enemy infantry strength at the end of the approach to contact. This situation is shown diagrammatically in Figure 1. The reader is referred to Kawara's paper [22] for further details of the model's development. It should be pointed out that this model also applies to the case of an amphibious landing and the determination of the optimal time-sequential allocation of the supporting fires of Naval ship guns.

Mathematically, the problem may be stated as the following.†

$$\underset{U}{\text{maximize}}\ \underset{V}{\text{minimize}} \left\{ \frac{x_1(t_f)}{y_1(t_f)} \right\},$$

with stopping rule: $t_f - T = 0$,

subject to: (battle dynamics)

$$\frac{dx_1}{dt} = -va_1x_1y_2, \qquad (1)$$

$$\frac{dx_2}{dt} = -(1-v)a_2y_2,$$

$$\frac{dy_1}{dt} = -ub_1y_1x_2,$$

$$\frac{dy_2}{dt} = -(1-u)b_2x_2,$$

with initial conditions

$$x_i(t=0) = x_i^o \quad \text{and} \quad y_i(t=0) = y_i^o \quad \text{for} \quad i = 1,2,$$

---

†We use capital letters to denote the closed-loop (or feedback) strategies (see [19]) of the players and the corresponding lower case letters to denote the corresponding strategic variables (see [4]). A strategic variable is the realization (or outcome) of a strategy. Thus, $u(t) = U(t,\underset{\sim}{x},\underset{\sim}{y})$ and $v(t) = V(t,\underset{\sim}{x},\underset{\sim}{y})$.

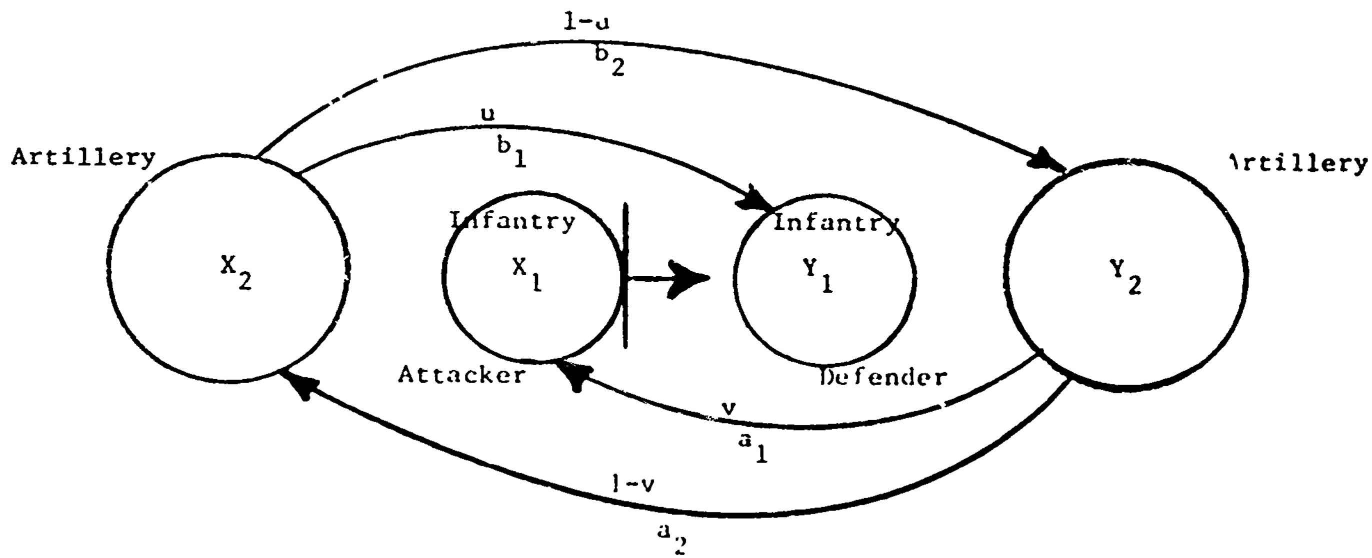

Figure 1. Diagram of Kawara's Fire-Support Differential Game.

and

$x_1, x_2, y_1, y_2 \geq 0$ (State Variable Inequality Constraints),

$0 \leq u, v \leq 1$ (Strategic Variable Inequality Constraints),

where

$x_1(t)$ is the number of X infantry (i.e. X_1) at time t,

$x_2(t)$ is the number of X artillery (i.e. X_2) at time t,

similarly for $y_1(t)$ and $y_2(t)$,

a_i is a constant (Lanchester) attrition-rate coefficient†

(reflecting the effectiveness of Y_2 fire against X_i),

similarly for b_i,

and

$u(v)$ is the fraction of $X(Y)$ artillery fire directed at opposing infantry forces.

We observe that for $T < +\infty$ it follows from the battle dynamics (1) that $x_1(t), y_1(t) > 0 \ \forall t \in [0,T]$. Thus, the only state variable inequality constraints (SVIC's) that must be considered are $x_2, y_2 \geq 0$.

Kawara's results and conclusions [22] have been discussed in Section 1 above.

3. Another Model for Optimal Fire-Support Allocation.

In this paper we will study a variation of Kawara's [22] fire-support differential game (1) just given. We will see that for this problem the structure of the optimal fire-support strategy for the attacker has a fundamentally different form than that for (1): the attacker must sometimes split his fire between the defender's primary and supporting units in order to "avoid overkill."

†See [10] (also [8], [9]) for methodology for the prediction of such coefficients from weapon system performance data.

Furthermore, the nature of this split in an optimal strategy depends on the allocation of the defender's supporting fires.

Let us again consider the attack of heterogeneous X forces against the static defense of heterogeneous Y forces. Each side is composed of primary units (or infantry) and fire support units (or artillery). The X infantry (denoted as X_1) launches an attack against the position held by the Y infantry (denoted as Y_1). Again, we will consider only the "approach to contact" phase of the battle. This pahse is the time from the initiation of the advance of the X_1 forces towards the Y_1 defensive position until the X_1 forces actually make contact with the enemy infantry. It is assumed that this time is fixed and known to both sides.

The X_1 forces begin their advance against the Y_1 forces from a distance and move towards the Y_1 position using "cover and concealment." The objective of the X_1 forces during the "approach to contact" is to close with the enemy position as rapidly as possible. Accordingly, small arms fire by the X_1 forces is held at a minimum or firing is done "on the move" to facilitate their rapid movement. It is not unreasonable, therefore, to assume that the effectiveness of X_1's fire "on the move" is negligible against Y_1. We assume, however, that the defensive Y_1 fire causes attrition to the advancing X_1 forces at a rate proportional to the product of the numbers of firers and targets. Two possible justifications of this are as follows: because of the movement (and intermittent concealment) of the X_1 forces and the distance involved, the Y_1 defenders either (1) fire into a constant (but moving) area without precise knowledge of the consequences of their fire or (2) when they do aim fire at X_1 targets, the time to acquire such a target is inversely proportional to the density of X_1 forces and much greater than the time to kill an acquired target. Under each of these sets of circumstances the assumed form of attrition has been hypotehsized to occur [11], [37].

During the "approach to contact," the fire-support units remain stationary. Each unit has the capability to deliver counterbattery fire against enemy artillery or "area fire" against the enemy's infantry. In other words, we assume that each side's fire support units fire into the (constant) area containing the enemy's infantry without feedback as to the destructiveness of this fire. On the other hand, the effectiveness of counterbattery fire is not symmetric with respect to the two combatants. We assume that the defender has the capability (for example, through the use of aerial observers) to sense when an enemy supporting unit has been destroyed so that fire may be immediately shifted to a new target† and that fire is uniformly distributed over the survivors.†† The attacker, however, either (1) does not have the capability to sense destruction of enemy fire support units accurately (and hence distributes his fire uniformly over the (constant) area occupied by the defender's fire support units) or (2) if he does have adequate fire assessment capability, then target acquisition times (which are inversely proportional to the density of the enemy's fire support units) are much larger than the time to destroy an acquired target. This leads to a Y_2 attrition rate proportional to the product of the numbers of X_2 firers and Y_2 targets [11], [37].

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†Alternatively, we may think that the attacker has massed so much artillery that X_2 targets are always easily acquired by Y_2 once an X_2 unit has been destroyed. Moreover, it will be assumed below that the initial X_2 force level is sufficiently large to guarantee that it is never driven to zero.

††This assumption is not essential for the structure of X_2's optimal fire support strategy. A similar structural result may be obtained when X_2's attrition is the same form as that for Y_2. We have made the above assumption, moreover, so that the resultant attrition model is most similar to Kawara's [22] but yet yields significantly different results for the attacker's fire support strategy.

It is the objective of each side to attain the most favorable infantry force ratio possible at the end of the "approach to contact" at which time the force separation between the opposing infantries is zero and artillery fires must be lifted from the enemy's infantry's position in order not to also kill friendly forces. Thus, the decision problem facing each side is to determine the "best" distribution of artillery fire between enemy infantry and artillery over time in order to maximize the infantry force ratio at the time of contact between the two infantry forces. This situation is shown diagrammatically in Figure 2.

The above assumptions lead to the following differential game with an attrition structure slightly different than that in Kawara's problem [22].

$$\underset{U}{\text{maximize}}\ \underset{V}{\text{minimize}} \left\{ \frac{x_1(t_f)}{y_1(t_f)} \right\},$$

with stopping rule: $t_f - T = 0,$

subject to: (battle dynamics)

$$\frac{dx_1}{dt} = -a_{11}x_1y_1 - va_{12}x_1y_2, \qquad (2)$$

$$\frac{dx_2}{dt} = -(1-v)a_2y_2,$$

$$\frac{dy_1}{dt} = -ub_1y_1x_2,$$

$$\frac{dy_2}{dt} = -(1-u)b_2y_2x_2,$$

with initial conditions

$$x_i(t=0) = x_i^o \quad \text{and} \quad y_i(t=0) = y_i^o \quad \text{for } i = 1,2,$$

and

$x_1, x_2, y_1, y_2 \geq 0$ (State Variable Inequality Constraints),

$0 \leq u, v \leq 1$ (Strategic Variable Inequality Constraints),

where all symbols are (essentially) the same as defined above for problem (1).

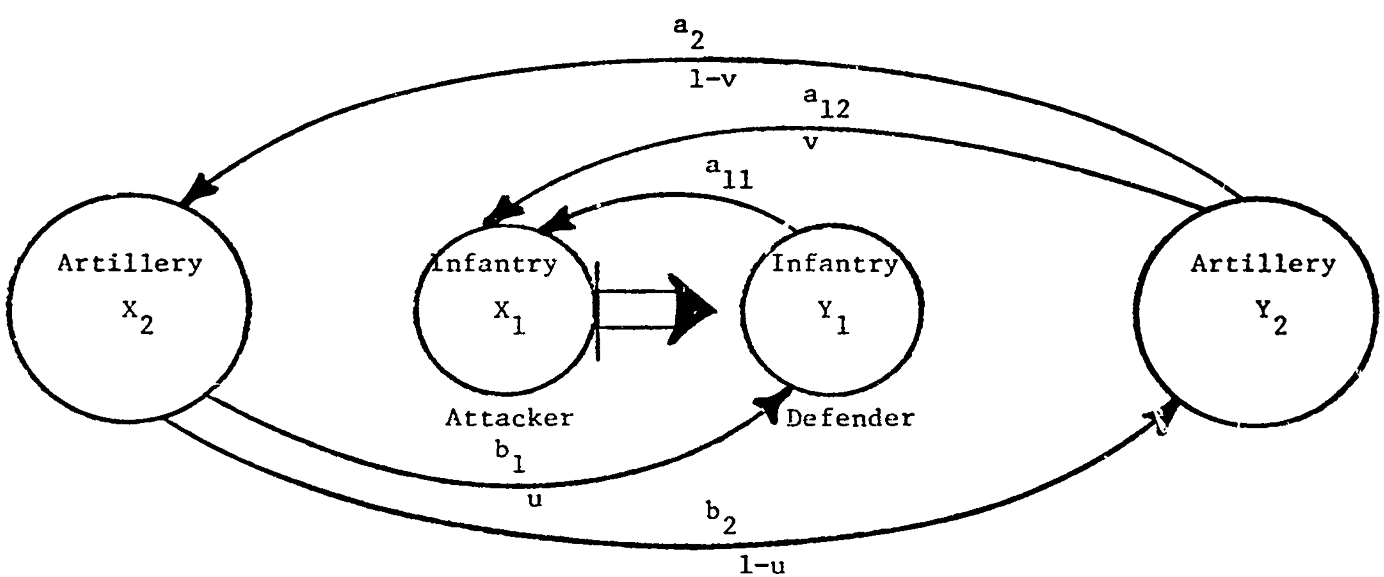

Figure 2. Diagram of Fire-Support Differential Game Studied in this Paper.

We observe that for $T < +\infty$ it follows from the battle dynamics (1) that $x_1(t)$, $y_1(t)$, and $y_2(t) > 0$ $\forall t \in [0,T]$. Thus the only SVIC that must be considered is $x_2 \geq 0$. However, let us assume that the force level of the attacker's artillery is never reduced to zero. In other words, we consider the special case in which x_2^o and T are such that $x_2(T) > 0$.

4. Characterization of Optimal Fire-Distribution Strategies for the Supporting Weapon Systems.

It should be clear that in (2) above we have $a_{11}, a_{12}, a_2, b_1, b_2 > 0$. Although the results of A. Friedman [14] concerning existence of value do not apply directly to our fire-support differential game (2), they do apply to a suitably modified version. If we were to consider a version of this problem with $\frac{dx_2}{dt} = -(1-v)a_2y_2 + r_2$ where $r_2 > 0$, then it may be shown (see pp. 210-230 of [14]) that this "modified" differential game has value and that a saddle point exists in pure strategies (see pp. 234-235 of [14]). We will now develop the basic necessary conditions of optimality for (2).

For $x_1, x_2, y_1, y_2 > 0$, the Hamiltonian for (2) is given by [12]

$$H(t,\underset{\sim}{x},\underset{\sim}{y},\underset{\sim}{p},\underset{\sim}{q},u,v) = -p_1(a_{11}x_1y_1 + va_{12}x_1y_2) - p_2a_2(1-v)y_2$$

$$- q_1ub_1y_1x_2 - q_2(1-u)b_2y_2x_2, \qquad (3)$$

where we have adopted the following correspondence between state and adjoint variables:

state variable	dual variable	
x_i	p_i	
y_i	q_i	for $i = 1,2$.

The adjoint system of differential equations for the dual variables is

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x_1} = a_{11}y_1p_1 + v^*a_{12}y_2p_1 \qquad \text{with} \quad p_1(T) = \frac{1}{y_1^f}, \tag{4}$$

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial x_2} = u^*b_1y_1q_1 + (1-u^*)b_2y_2q_2 \qquad \text{with} \quad p_2(T) = 0, \tag{5}$$

$$\frac{dq_1}{dt} = -\frac{\partial H}{\partial y_1} = a_{11}x_1p_1 + u^*b_1x_2q_1 \qquad \text{with} \quad q_2(T) = -\frac{x_1^f}{(y_1^f)^2}, \tag{6}$$

$$\frac{dq_2}{dt} = -\frac{\partial H}{\partial y_2} = v^*a_{12}x_1p_1 + (1-v^*)a_2p_2 + (1-u^*)b_2x_2q_2 \quad \text{with} \quad q_2(T) = 0. \tag{7}$$

The results of Berkovitz [3] say that H, $\underset{\sim}{p}(t)$, and $\underset{\sim}{q}(t)$ are continuous functions of time except possibly at manifolds of discontinuity of both U^* and V^* (see Section 4.3 below).

When $x_1, x_2, y_1, y_2 > 0$, the extremal strategic-variable pair, denoted as (u^*, v^*), is determined by the max-min principle. Hence, we consider

$$\underset{0 \le u \le 1}{\text{maximize}} \; \underset{0 \le v \le 1}{\text{minimize}} \; H(t, \underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{p}, \underset{\sim}{q}, u, v),$$

so that

$$u^*(t) = \begin{cases} 1 & \text{for} \quad S_u(t) > 0, \\ 0 & \text{for} \quad S_u(t) < 0, \end{cases} \tag{8}$$

where the U-switching function, $S_u(t)$, is given by

$$S_u(t) = b_1(-q_1)y_1 - b_2(-q_2)y_2, \tag{9}$$

and

$$v^*(t) = \begin{cases} 1 & \text{for} \quad S_v(t) > 0, \\ 0 & \text{for} \quad S_v(t) < 0, \end{cases} \tag{10}$$

where the V-switching function, $S_v(t)$, is given by

$$S_v(t) = a_{11}p_1x_1 - a_2p_2. \tag{11}$$

It is readily shown that

$$p_1(t)x_1(t) = \text{constant} = p_1(T)x_1(T) = \frac{x_1^f}{y_1^f}, \tag{12}$$

$$\frac{d}{dt}(q_1y_1) = a_{11}\frac{x_1^f}{y_1^f}y_1(t) > 0, \tag{13}$$

and

$$\frac{dS_v}{dt} = -a_2(1-u^*)S_u(t) - a_2b_1q_1y_1. \tag{14}$$

We must further investigate the possibility of singular subarcs (see [31] or Chapter 8 of [12]). Let us first show that it is impossible to have a V-singular subarc. In other words, $v^*(t)$ must be 0 or 1 almost everywhere in time. The impossibility of a V-singular subarc is established by showing that $\frac{ds_v}{dt} > 0$ for all $t \varepsilon [0,T]$. It is clear that

$$(1-u^*)S_u(t) \leq 0 \quad \text{for all} \quad t \in [0,T]. \tag{15}$$

Considering (13) and the fact that $q_1(T)y_1(T) = -\frac{x_1^f}{y_1^f} < 0$, we see that $q_1(t)y_1(t) < 0$ for all $t \in [0,T]$, whence follows the assertion via (14).

It is possible, however, to have a U-singular subarc on which $\frac{\partial H}{\partial u} = 0$ (or, equivalently, $S_u(t) = 0$) for a finite interval of time. There are two cases to be considered: (1) $v^* = 1$ and (2) $v^* = 0$.

4.1. U-Singular Subarc on Which $V^* = 1$.

When $v^* = 1$, it is readily computed that

$$\frac{dS_u}{dt} = -\left(\frac{x_1^f}{y_1^f}\right)(a_{11}b_1y_1-a_{12}b_2y_2), \tag{16}$$

and

$$\frac{d^2 S_u}{dt^2} = -\left(\frac{x_1^f}{y_1^f}\right) x_2 \{(a_{11} b_1 y_1) u^* b_1 - (a_{12} b_2 y_2)(1-u^*) b_2\}. \tag{17}$$

Considering (9), the requirement that $\frac{\partial H}{\partial u} = 0$ yields the <u>first condition for a U-singular subarc with $V^* = 1$</u>

$$b_1 q_1 y_1 = b_2 q_2 y_2. \tag{18}$$

Considering (16) and (18), the requirement that $\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) = 0$ on a singular subarc on which $\frac{\partial H}{\partial u} = 0$ for a finite interval of time yields the <u>second condition for a U-singular subarc with $V^* = 1$</u>

$$a_{11} b_1 y_1 = a_{12} b_2 y_2 \tag{19}$$

On a subarc on which the first and second conditions for a singular subarc hold we additionally require that $\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) = 0$ so that (17) yields the <u>singular strategic variable value</u> required to keep the system on the singular subarc

$$u^*(t) = \frac{b_2}{b_1 + b_2} \tag{20}$$

Checking the generalized Legendre-Clebsch condition† [23], [24] $\frac{\partial}{\partial u}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right)\right\} \geq 0$, we find that on a subarc on which (18) and (19) hold we have

$$\frac{\partial}{\partial u}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right)\right\} = \left(\frac{x_1^f}{y_1^f}\right)(x_2)^2\{a_{11}(b_1)^2 y_1 + a_{12}(b_2)^2 y_2\} > 0.$$

We may write the equation of the U-singular surface (see p. 683 of [31] as

$$\frac{y_1}{y_2} = \frac{a_{12} b_2}{a_{11} b_1} \quad \text{for} \quad v^* = 1. \tag{21}$$

---

†This is a necessary condition for optimality. R. Isaacs [21] gives an equivalent condition (see [13]).

4.2. U-Singular Subarc on Which $V^* = 0$.

When $v^* = 0$, it is readily computed that

$$\frac{dS_u}{dt} = -\frac{x_1^f}{y_1^f}(a_{11}b_1y_1 - a_{12}b_2y_2) - b_2y_2S_v(t), \tag{22}$$

and

$$\frac{d^2S_u}{dt^2} = -u^*b_1x_2\frac{dS_u}{dt} + a_2b_2y_2\{-u^*S_u(t)+b_2q_2y_2+p_2x_2[u^*b_1-(1-u^*)b_2]\}, \tag{23}$$

so that the first and second conditions for a U-singular subarc with $V^* = 0$ are, respectively, (18) and

$$a_{11}b_1y_1 = a_{12}b_2y_2 + b_2y_2\left(\frac{y_1^f}{x_1^f}\right)\{-S_v(t)\}. \tag{24}$$

It should be noted (see [18]) that the above singular surface exists in $\underset{\sim}{x} - \underset{\sim}{p}$ space. It is convenient to write

$$\frac{y_1}{y_2} = \frac{a_{12}b_2}{a_{11}b_1} + \frac{b_2y_1^f}{a_{11}b_1x_1^f}\{-S_v(t)\} \quad \text{for} \quad v^* = 0. \tag{25}$$

The singular strategic variable value is given by

$$u^*(t) = \left(\frac{b_2}{b_1+b_2}\right)\left(1 - \frac{q_2y_2}{p_2x_2}\right). \tag{26}$$

The requirement that $u^* \leq 1$ yields that on a U-singular subarc with $V^* = 0$ we must have

$$b_2(-q_2)y_2 \leq b_1p_2x_2. \tag{27}$$

It is readily checked that the generalized Legendre-Clebsch condition is satisfied.

4.3. Discontinuity of Adjoint Variables Across Manifold of Discontinuity of Both U^* and V^*.

It is convenient to introduce the backwards time τ defined by

$$\tau = T - t. \tag{28}$$

From (20) and (26), we see that $u^*(\tau)$ must change, in general, discontinuously from $b_2/(b_1+b_2)$ to $b_2/(b_1+b_2)(1-q_2y_2/(p_2x_2))$ whenever $v^*(\tau)$ changes from 1 to 0. Let us consider the totality of trajectories on which this happens. The locus of points in the $t,\underset{\sim}{x},\underset{\sim}{y}$ - space for such simultaneous switches is then a manifold of discontinuity of both U^* and V^*. Across such a manifold the adjoint variables need not be continuous (see [3]).

Let $\tau_v = \tau_v(\underset{\sim}{x},\underset{\sim}{y})$ denote the backwards time at which $v^*(\tau)$ changes from 1 to 0. For future purposes, it will be convenient to consider a simultaneous switch with u^* changing from the singular control $b_2/(b_1+b_2)$ to 1. Then the manifold of discontinuity of both U^* and V^* is given by

$$F(t,\underset{\sim}{x},\underset{\sim}{y}) = t - T + \tau_v(\underset{\sim}{x},\underset{\sim}{y}) = 0,$$

and (29)

$$G(\underset{\sim}{y}) = a_{11}b_1y_1 - a_{12}b_2y_2 = 0.$$

Across the manifold of discontinuity, we have

$$\underset{\sim}{p}^T(\tau_v^+) = \underset{\sim}{p}^T(\tau_v^-) - \rho\,\frac{\partial F}{\partial \underset{\sim}{x}} - \sigma\,\frac{\partial G}{\partial \underset{\sim}{x}}\,,$$

$$\underset{\sim}{q}^T(\tau_v^+) = \underset{\sim}{q}^T(\tau_v^-) - \rho\,\frac{\partial F}{\partial \underset{\sim}{y}} - \sigma\,\frac{\partial G}{\partial \underset{\sim}{y}}\,,$$

and

$$H(\tau_v^+) = H(\tau_v^-) + \rho\,\frac{\partial F}{\partial t} + \sigma\,\frac{\partial G}{\partial t}\,,$$

or

$$p^T(\tau_v^+) = p^T(\tau_v^-) - \rho\,\frac{\partial \tau_v}{\partial \underset{\sim}{x}}\,, \tag{30}$$

$$(-q_1(\tau_v^+)) = (-q_1(\tau_v^-)) + \rho\,\frac{\partial \tau_v}{\partial y_1} + \sigma a_{11}b_1\,, \tag{31}$$

$$(-q_2(\tau_v^+)) = (-q_2(\tau_v^-)) + \rho\,\frac{\partial \tau_v}{\partial y_2} - \sigma a_{12}b_2\,, \tag{32}$$

and

$$H(\tau_v^+) = H(\tau_v^-) + \rho. \tag{33}$$

Considering (9) and (11), it is readily shown that

$$S_u(\tau_v^+) = \sigma\{a_{11}(b_1)^2y_1 + a_{12}(b_2)^2y_2\} + \rho(b_1y_1 \frac{\partial\tau_v}{\partial y_1} - b_2y_2 \frac{\partial\tau_v}{\partial y_2}), \tag{34}$$

and

$$S_v(\tau_v^+) = -\rho(a_{12}x_1 \frac{\partial\tau_v}{\partial x_1} - a_2 \frac{\partial\tau_v}{\partial x_2}). \tag{35}$$

Recalling that $u^*(\tau_v^-) = b_2/(b_1+b_2)$, $u^*(\tau_v^+) = 1$, $v^*(\tau_v^-) = 1$, and $v^*(\tau_v^+) = 0$, we may substitute (30) through (32) into (33) to obtain for $a_{11}x_1y_1 \frac{\partial\tau_v}{\partial x_1} + a_2y_2 \frac{\partial\tau_v}{\partial x_2} + b_1y_1x_2 \frac{\partial\tau_v}{\partial y_1} \neq 1$

$$\rho = \frac{a_{11}(b_1)^2y_1x_2\sigma}{\left(1-a_{11}x_1y_1 \frac{\partial\tau_v}{\partial x_1} - a_2y_2 \frac{\partial\tau_v}{\partial x_2} - b_1y_1x_2 \frac{\partial\tau_v}{\partial y_1}\right)}. \tag{36}$$

Then we may write

$$S_u(\tau_v^+) = \sigma\Bigg\{a_{11}(b_1)^2y_1 + a_{12}(b_2)^2y_2 + \frac{a_{11}(b_1)^2y_1x_2(b_1y_1 \frac{\partial\tau_v}{\partial y_1} - b_2y_2 \frac{\partial\tau_v}{\partial y_2})}{\left(1-a_{11}x_1y_1 \frac{\partial\tau_v}{\partial x_1} - a_2y_2 \frac{\partial\tau_v}{\partial x_2} - b_1y_1x_2 \frac{\partial\tau_v}{\partial y_1}\right)}\Bigg\}, \tag{37}$$

and

$$S_v(\tau_v^+) = \frac{-a_{11}(b_1)^2y_1x_2(a_{12}x_1 \frac{\partial\tau_v}{\partial x_1} - a_2 \frac{\partial\tau_v}{\partial x_2})\sigma}{\left(1-a_{11}x_1y_1 \frac{\partial\tau_v}{\partial x_1} - a_2y_2 \frac{\partial\tau_v}{\partial x_2} - b_1y_1x_2 \frac{\partial\tau_v}{\partial y_1}\right)}. \tag{38}$$

5. <u>Synthesis of Extremal Strategic-Variable Pair.</u>

By the synthesis of the extremal strategic variable pair we mean the explicit determination (using the basic necessary conditions of optimality) of the time history of the extremal strategic variable pair $(u^*,v^*)^\dagger$ from initial to terminal time (see [21] and also [31]-[33]). The basic idea is to trace extremals backwards from the terminal manifold (where boundary conditions for the adjoint variables are known) in such a way to guarantee the satisfaction of the initial conditions. Thus, it is convenient to introduce the backwards time τ defined by (28).

5.1. Extremal Transitions in Strategic Variables.

It seems appropriate to examine what are the possible transitions (or changes) in each strategic variable as we work backwards from the end (i.e. as τ increases). It has been shown above that $\frac{\partial S_v}{d\tau} < 0$ for all $\tau \in [0,T]$. Considering the boundary conditions (4) and (5) for the adjoint variables, it follows that $S_v(\tau=0) > 0$. Thus

$$v^*(\tau) = \begin{cases} 1 & \text{for } 0 \leq \tau \leq \tau_v, \\ 0 & \text{for } \tau_v < \tau. \end{cases} \tag{39}$$

It will be convenient to refer to that phase of the planning horizon during which $v^*(t) = 0$ as V-Phase I (i.e. $0 \leq t < T - \tau_v$) and to that during which $v^*(t) = 1$ as V-Phase II.

Extremal transitions in u^* for increasing τ as shown in Figure 3. Thus, this figure shows what changes we might expect to observe in u^* as we follow an extremal backwards from the end of the planning horizon at $\tau = 0$. During V-Phase II when $v^* = 1$, $\frac{dS_u}{d\tau} = \left(\frac{x_1^f}{y_1^f}\right)(a_{11}b_1y_1 - a_{12}b_2y_2)$ with $S_u(\tau=0) = b_1x_1^f/y_1^f > 0$. When $u^* = 0$, then $\frac{d}{d\tau}\left(\frac{y_1}{y_2}\right) < 0$. During V-Phase I when $v^* = 0$, $\frac{dS_u}{d\tau} = \left(\frac{x_1^f}{y_1^f}\right)(a_{11}b_1y_1 - a_{12}b_2y_2) + b_2y_2S_v(\tau)$. During both phases, the singular subarc may be exited with either $u^* = 0$ or $u^* = 1$. Once u^* becomes 0, it remains this way. The above statements will be further justified below.

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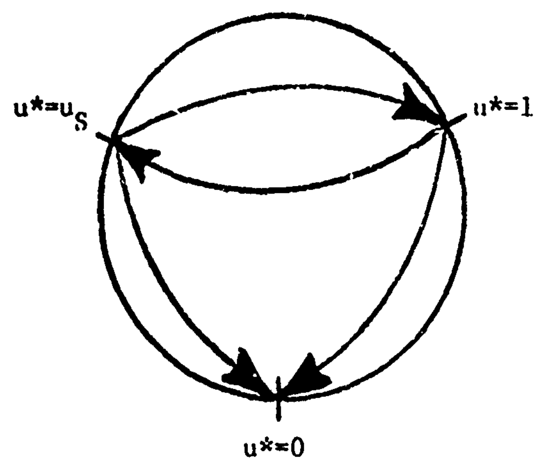
†It should be kept in mind that, for example, $u^*(t) = U^*(t,\underset{\sim}{x},\underset{\sim}{y})$.



Note:

$$u_S = \begin{cases} \left(\dfrac{b_2}{b_1+b_2}\right)\left(1 - \dfrac{q_2y_2}{p_2x_2}\right) & \text{for } v^* = 0, \\ \\ \left(\dfrac{b_2}{b_1+b_2}\right) & \text{for } v^* = 1. \end{cases}$$

Figure 3. Extremal Transitions in u^* for Increasing τ.

5.2. Extremal Synthesis for $\tau_u^* < \tau_v$.

From the above we have

$$S_u(\tau=0) = b_1 \frac{x_1^f}{y_1^f} > 0, \tag{40}$$

so that by (8) we have

$$u^*(\tau) = 1 \quad \text{for} \quad 0 \le \tau \le \tau_u, \tag{41}$$

where τ_u is the smallest zero of the equation $S_u(\tau=\tau_u) = 0$. If the U-singular subarc is reached in V-Phase II (see Section 4.1. above), then let us denote the backwards switching time at which u^* changes from 1 to $b_2/(b_1+b_2)$ as τ_u^*. Clearly, it is necessary that $\tau_u^* < \tau_v$ for this singular subarc to appear in the solution. Thus, in general, there are two cases to be considered:

(1) $\tau_u^* < \tau_v$,

and (2) $\tau_u^* \ge \tau_v$.

In this paper we will consider only the former case, with the latter one following along the same general lines of development. We therefore assume that a_{11}, a_{12}, a_2, b_1, b_2, x_1^f, x_2^f, y_1^f, and y_2^f are such that $\tau_u^* < \tau_v$. We will give numerical results for this case below. Moreover, in all our numerical computations we have only encountered this case.

5.2.1. Extremals Near the Terminal Manifold.

Recalling (16), we see that $\frac{dS_u}{d\tau} > 0$ (<0) if and only if $\frac{y_1}{y_2} > (<) \frac{a_{12}b_2}{a_{11}b_1}$. Considering (40), it is clear that $S_u(\tau) > 0$ for $v^* = 1$ when $\frac{y_1^f}{y_2^f} > \frac{a_{12}b_2}{a_{11}b_1}$. However, $S_u(\tau)$ may change sign when $\frac{y_1^f}{y_2^f} < \frac{a_{12}b_2}{a_{11}b_1}$. The U-singular subarc occurs when both $S_u(\tau=\tau_u^*) = 0$ and $a_{11}b_1y_1 = a_{12}b_2y_2$ at $\tau = \tau_u^*$. Thus, τ_u^* is the smallest root of

$$-\frac{1}{b_1x_2^f} + \tau_u^* + \left(\frac{1}{b_1x_2^f} - \frac{1}{a_{11}y_1^f}\right)e^{-b_1x_2^f\tau_u^*} = 0. \qquad (42)$$

If y_1^f is given, then $S_u(\tau=\tau_u^*) = 0$ and $a_{11}b_1y_1 = a_{12}b_2y_2$ may be combined to yield the value of y_2^f required in order to reach the U-singular subarc (denoted as y_2^f*). Thus, for $a_{11}y_1^f \neq b_1x_2^f$ we have $y_2^f = \frac{b_1}{b_2}\frac{(a_{11}y_1^f-b_1x_2^f)}{a_{12}(1-b_1x_2^f\tau_u^*)}$. (Other results are given below in Table I.) We denote the corresponding ratio of y_1^f to y_2^f* as $\left(\frac{y_1^f}{y_2^f}\right)^*$.

When $S_u(\tau=\tau_u) = 0$ with $a_{11}b_1y_1 < a_{12}b_2y_2$, it follows that τ_u is the smallest root of the transcendental equation

$$\left(b_1 - \frac{a_{11}y_1^f}{x_2^f}\right) - a_{12}b_2y_2^f\tau_u + \frac{a_{11}y_1^f}{x_2^f}e^{b_1x_2^f\tau_u} = 0. \qquad (43)$$

It may be shown that $\frac{\partial\tau_u}{\partial r} > 0$ where $r = y_1^f/y_2^f$. This latter result is useful in proving the following:

THEOREM 1: Assume that $\tau_v > \tau_u^*$. Then, $u^*(\tau) = 1$ on any extremal as long as $v^*(\tau) = 1$ for $\frac{y_1^f}{y_2^f} > \left(\frac{y_1^f}{y_2^f}\right)^*$.

PROOF: The proof is by contradiction. Let $r = y_1^f/y_2^f$.

(a) Assume that we could have a switch in $u^*(\tau)$ (with $v^*(\tau) = 1$) for $\frac{y_1^f}{y_2^f} > \left(\frac{y_1^f}{y_2^f}\right)^*$. In other words, we can find τ_u such that $S_u(\tau=\tau_u) = 0$ with

$$S_u(\tau) > 0 \quad \text{for} \quad 0 \le \tau < \tau_u, \qquad (44)$$

for $\frac{y_1^f}{y_2^f} > \left(\frac{y_1^f}{y_2^f}\right)^*$.

(b) Consider $\tilde{r} = \frac{y_1^f}{y_2^f} = \left(\frac{y_1^f}{y_2^f}\right)^* + \varepsilon$ with $\varepsilon > 0$ and such that $\tau_u < \tau_v$. Then it may be shown that $\frac{\partial \tau_u}{\partial r} > 0$. In particular $\left.\frac{\partial \tau_u}{\partial r}\right|_{r=\left(\frac{y_1^f}{y_2^f}\right)^*} > 0$. This implies, however, that $\tau_u > \tau_u^*$ for $r = \tilde{r}$.

(c) Observe that $\begin{cases} u^* = 1 \\ v^* = 1 \end{cases}$ for $0 \le \tau \le \tau_u < \tau_v$ so that $y_1/y_2 = y_1^f/y_2^f \, e^{b_1 x_2^f \tau}$. Hence,

$$\frac{y_1}{y_2}(\tau=\tau_u) > \frac{a_{12}b_2}{a_{11}b_1} \quad \text{for} \quad r = \tilde{r}, \tag{45}$$

since then $\tau_u(r=\tilde{r}) > \tau_u^*$. It has been shown above that $\frac{dS_u}{d\tau} > 0$ for $y_1/y_2 > a_{12}b_2/(a_{11}b_1)$. Thus, (45) implies that $\frac{dS_u}{d\tau}(\tau=\tau_u) > 0$, and hence

$$0 = S_u(\tau=\tau_u) > S_u(\tau) \quad \text{for} \quad \tau \in (\tau_u-\delta, \tau_u). \tag{46}$$

This last statement (46) is a contradiction to (44), and the theorem is proved. Q.E.D.

Other results are obtained in a similar fashion.

5.2.2. Field Construction.

For a given set of terminal values x_1^f, x_2^f, y_1^f, and y_2^f an extremal may be traced backwards from the terminal manifold by a backwards integration of the state and adjoint equations combined with (8) and (10) (also (20) or (26)). By varying these terminal values the entire field of extremals (see p. 128 of [12]) may be obtained.

The various types of extremals that may occur in the field of extremals are shown in Figure 4. This figure is representative of all our numerical results for $\tau_u^* < \tau_v$ (see Section 5.2.3 below). Pertinent information concerning each type of extremal is given in Table I.

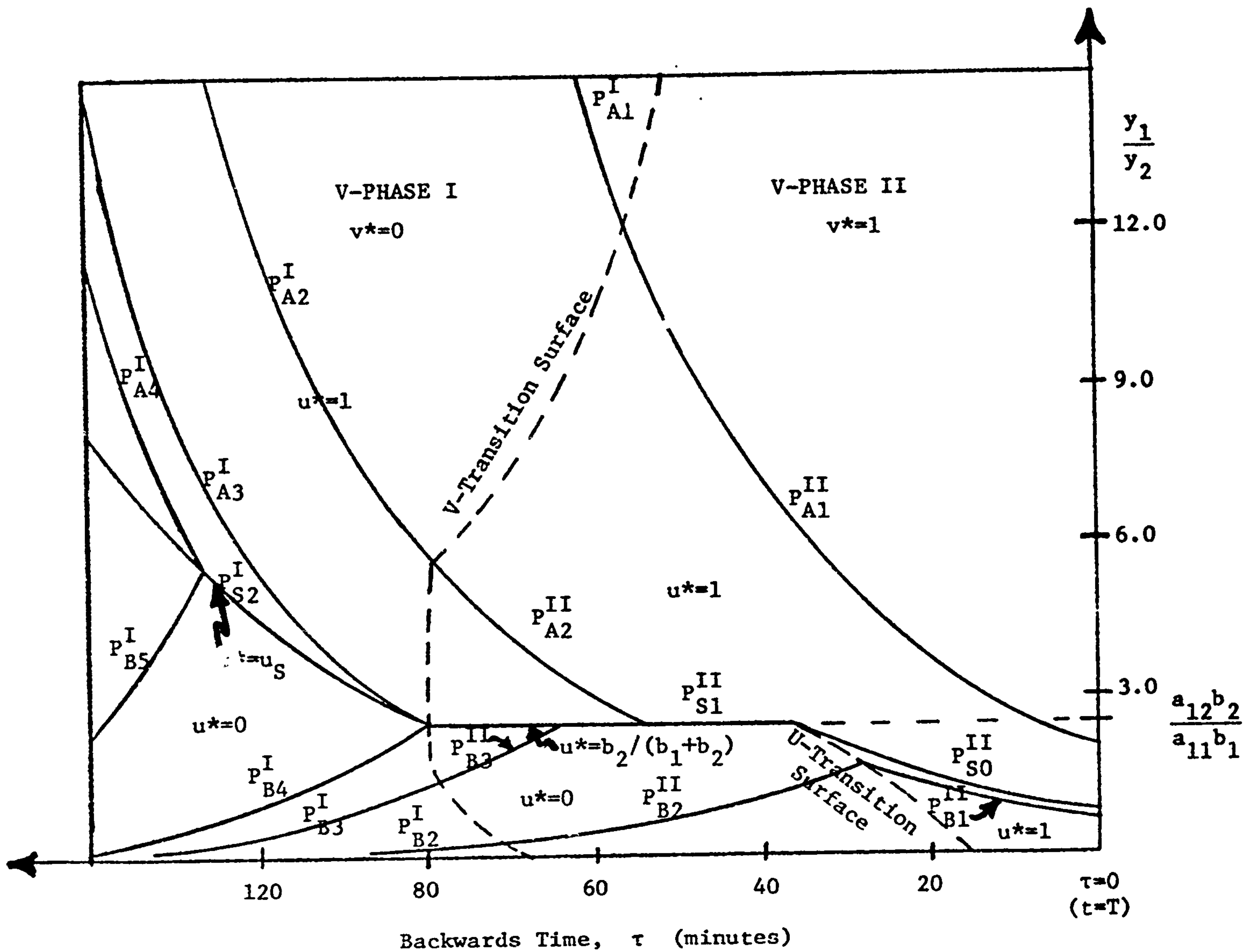


Figure 4. Identification of Various Types of Extremals for Which Information is Given in Table I.

Table I. Extremal Trajectories for Fire Support Problem with $\tau_u^* < \tau_v$.

1. $P_{SO}^{II}: \begin{cases} u^*(\tau) = 1 \\ v^*(\tau) = 1 \end{cases}$ for $0 \le \tau \le \tau_u^*$ with $\dfrac{y_1^f}{y_2^f} = \left(\dfrac{y_1^f}{y_2^f}\right)^*$

τ_u^* is the smallest positive root of

$$-\frac{1}{b_1x_2^f} + \tau_u^* + \left(\frac{1}{b_1x_2^f} - \frac{1}{a_{11}y_1^f}\right)e^{-b_1x_2^f\tau_u^*} = 0,$$

with the following bounds established:

$$\text{for} \quad a_{11}y_1^f > b_1x_2^f: \quad \frac{1}{a_{11}y_1^f} < \tau_u^* < \frac{1}{b_1x_2^f},$$

$$\text{for} \quad a_{11}y_1^f = b_1x_2^f: \quad \tau_u^* = \frac{1}{b_1x_2^f},$$

$$\text{for} \quad a_{11}y_1^f < b_1x_2^f: \quad \frac{1}{b_1x_2^f} < \tau_u^* < \frac{1}{a_{11}y_1^f}.$$

$$\text{for} \quad a_{11}y_1^f \ne b_1x_2^f: \quad y_2^{f*} = \frac{b_1}{b_2}\,\frac{(a_{11}y_1^f - b_1x_2^f)}{a_{12}(1-b_1x_2^f\tau_u^*)},$$

$$\text{for} \quad a_{11}y_1^f = b_1x_2^f: \quad y_2^{f*} = \frac{a_{11}b_1y_1^f}{a_{12}b_2}\,e.$$

Also

$$S_v(\tau) = a_2b_1\left(\frac{x_1^f}{y_1^f}\right)\left\{\left[\frac{a_{12}}{a_2b_1} + \frac{a_{11}y_1^f}{(b_1x_2^f)^2}\right] + \left(\frac{a_{11}y_1^f}{b_1x_2^f} - 1\right)\tau - \frac{a_{11}y_1^f}{(b_1x_2^f)^2}\,e^{b_1x_2^f\tau}\right\}.$$

Table I. (cont.) - 1

1. P_{SO}^{II}: (concluded)

Let $S_v(\tau=\tau_u^*)$. Also, on P_{SO}^{II} we have

$$x_1(\tau) = x_1^f \exp\{a_{12}y_2^f\tau + \frac{a_{11}y_1^f}{b_1x_2^f}(e^{b_1x_2^f\tau}-1)\} ,$$

$$x_2(\tau) = x_2^f ,$$

$$y_1(\tau) = y_1^f e^{b_1x_2^f\tau} ,$$

$$y_2(\tau) = y_2^f ,$$

and

$$p_1(\tau) = \frac{1}{y_1^f}\exp\{-[a_{12}y_2^f\tau + \frac{a_{11}y_1^f}{b_1x_2^f}(e^{b_1x_2^f\tau}-1)]\} ,$$

$$p_2(\tau) = b_1\left(\frac{x_1^f}{y_1^f}\right)\{\left(1 - \frac{a_{11}y_1^f}{b_1x_2^f}\right)\tau + \frac{a_{11}y_1^f}{(b_1x_2^f)^2}(e^{b_1x_2^f\tau}-1)\} ,$$

$$q_1(\tau) = \left(\frac{x_1^f}{y_1^f}\right)\{-\frac{a_{11}}{b_1x_2^f} + \left(\frac{a_{11}}{b_1x_2^f} - \frac{1}{y_1^f}\right)e^{-b_1x_2^f\tau}\} ,$$

$$q_2(\tau) = -a_{12}\left(\frac{x_1^f}{y_1^f}\right)\tau .$$

2. P_{S1}^{II}: $\begin{cases} u^*(\tau) = b_2/(b_1+b_2) \\ v^*(\tau) = 1 \end{cases}$ for $\tau_u^* \le \tau \le \tau_v^*$

where τ_u^* is determined in 1. On P_{S1}^{II} we have

$$S_u(\tau) = 0,$$

$$\text{and} \qquad a_{11}b_1y_1 = a_{12}b_2y_2.$$

τ_v^* is the smallest positive root of $S_v(\tau=\tau_v^*) = 0$, where

$$S_v(\tau) = S_v^{u*} + a_2b_1y_1^{u*}\left(\frac{x_1^f}{y_1^f}\right)\left\{\frac{a_{11}}{(\theta x_2^f)^2} + \left[q_1^{u*}\left(\frac{y_1^f}{x_1^f}\right) + \frac{a_{11}}{\theta x_2^f}\right](\tau-\tau_u^*) - \frac{a_{11}}{(\theta x_2^f)^2}\exp[\theta x_2^f(\tau-\tau_u^*)]\right\},$$

with $\theta = b_1b_2/(b_1+b_2)$. An upper bound on τ_v^* is given by

$$\hat{\tau}_v^* = a_{12}/(a_2b_1).$$

Also, on P_{S1}^{II} we have

$$x_1(\tau) = x_1^{u*}\exp\left\{\left[\frac{a_{11}y_1^{u*}+a_{12}y_2^f}{\theta x_2^f}\right]\left[e^{\theta x_2^f(\tau-\tau_u^*)}-1\right]\right\},$$

$$x_2(\tau) = x_2^f,$$

$$y_1(\tau) = y_1^{u*}e^{\theta x_2^f(\tau-\tau_u^*)},$$

$$y_2(\tau) = y_2^f e^{\theta x_2^f(\tau-\tau_u^*)},$$

and

$$p_1(\tau) = p_1^{u*}\exp\left\{-\left[\frac{a_{11}y_1^{u*}+a_{12}y_2^f}{\theta x_2^f}\right]\left[e^{\theta x_2^f(\tau-\tau_u^*)}-1\right]\right\} \quad \text{with} \quad p_1^{u*} = \frac{1}{x_1^{u*}}\left(\frac{x_1^f}{y_1^f}\right),$$

$$p_2(\tau) = p_2^{u*} - b_1y_1^{u*}\left(\frac{x_1^f}{y_1^f}\right)\left\{\frac{a_{11}}{(\theta x_2^f)^2} + \left[q_1^{u*}\left(\frac{y_1^f}{x_1^f}\right) + \frac{a_{11}}{\theta x_2^f}\right](\tau-\tau_u^*) - \frac{a_{11}}{(\theta x_2^f)^2}e^{\theta x_2^f(\tau-\tau_u^*)}\right\},$$

$$q_1(\tau) = q_1^{u*}e^{-\theta x_2^f(\tau-\tau_u^*)} - \left(\frac{x_1^f}{y_1^f}\right)\frac{a_{11}}{\theta x_2^f}\left\{1 - e^{-\theta x_2^f(\tau-\tau_u^*)}\right\},$$

$$q_2(\tau) = q_2^{u*}e^{-\theta x_2^f(\tau-\tau_u^*)} - \left(\frac{x_1^f}{y_1^f}\right)\frac{a_{12}}{\theta x_2^f}\left\{1 - e^{-\theta x_2^f(\tau-\tau_u^*)}\right\}.$$

3. P^{II}_{A1}: $\begin{cases} u^*(\tau) = 1 \\ v^*(\tau) = 1 \end{cases}$ for $0 \le \tau \le \tau_v$ with $\frac{y_1^f}{y_2^f} > \left(\frac{y_1^f}{y_2^f}\right)^*$

τ_v is the smallest positive root of $S_v(\tau=\tau_v) = 0$, where $S_v(\tau)$ is given in 1. An upper bound on τ_v is given by

$$\hat{\tau}_v = \frac{a_{12}}{a_2 b_1} .$$

It has been shown that $S_u(\tau) > 0$ for $0 \le \tau \le \tau_v$. The solutions to the state and adjoint equations are the same as those for P^{II}_{S0} given above.

4. P^{I}_{A1}: $\begin{cases} u^*(\tau) = 1 \\ v^*(\tau) = 0 \end{cases}$ for $\tau_v \le \tau \le \tau_u^I$

We have that $S_v(\tau) < 0$ for $\tau > \tau_v$ and that

$$\frac{dS_u}{d\tau}(\tau) = b_2 y_2^f S_v(\tau) + \left(\frac{x_1^f}{y_1^f}\right)(a_{11}b_1y_1(\tau) - a_{12}b_2y_2^f).$$

Also, on P^{I}_{A1} we have

$$\frac{dx_1}{d\tau} = a_{11}x_1y_1 \quad \text{with} \quad x_1(\tau=\tau_v) = x_1^v,$$

$$x_2(\tau) = x_2^v + a_2y_2^f(\tau-\tau_v),$$

$$y_1(\tau) = y_1^v \exp\{b_1x_2^v(\tau-\tau_v) + \frac{a_2b_1y_2^f}{2}(\tau-\tau_v)^2\},$$

$$y_2(\tau) = y_2^f,$$

and

$$\frac{dp_1}{d\tau} = -a_{11}y_1p_1 \quad \text{with} \quad p_1(\tau=\tau_v) = p_1^v,$$

$$\frac{dp_2}{d\tau} = -b_1y_1q_1 \quad \text{with} \quad p_2(\tau=\tau_v) = p_2^v,$$

$$\frac{dq_1}{d\tau} = -a_{11}\frac{x_1^f}{y_1^f} - b_1x_2q_1 \quad \text{with} \quad q_1(\tau=\tau_v) = q_1^v,$$

$$\frac{dq_2}{d\tau} = -a_2p_2 \quad \text{with} \quad q_2(\tau=\tau_v) = q_2^v.$$

We have not been able to develop solutions in terms of "elementary" functions to the equations for x_1, p_1, p_2, q_1, and q_2.

Table I. (cont.) - 4

5. P_{A2}^{II}: $\begin{cases} u^*(\tau) = 1 \\ v^*(\tau) = 1 \end{cases}$ for $\tau_{SL}^{II} \le \tau \le \tau_v$

τ_v is the smallest positive root of $S_v(\tau=\tau_v) = 0$, where

$$S_v(\tau) = S_v^{SLII} + a_2 b_1 y_1^{SLII}\left(\frac{x_1^f}{y_1^f}\right)\left\{\frac{a_{11}}{(b_1 x_2^f)^2} + \left[q_1^{SLII}\left(\frac{y_1^f}{x_1^f}\right) + \frac{a_{11}}{b_1 x_2^f}\right](\tau-\tau_{SL}^{II}) - \frac{a_{11}}{(b_1 x_2^f)^2} e^{b_1 x_2^f(\tau-\tau_{SL}^{II})}\right\}.$$

Again, an upper bound on τ_v is given by $a_{12}/(a_2 b_1)$. It has been shown that $S_u(\tau) > 0$ for $\tau_{SL}^{II} < \tau \le \tau_v$. Also, on P_{A2}^{II} we have

$$x_1(\tau) = x_1^{SLII} \exp\left\{a_{12} y_2^{SLII}(\tau-\tau_{SL}^{II}) + \frac{a_{11} y_1^{SLII}}{b_1 x_2^f}\left[e^{b_1 x_2^f(\tau-\tau_{SL}^{II})} - 1\right]\right\},$$

$$x_2(\tau) = x_2^f,$$

$$y_1(\tau) = y_1^{SLII} e^{b_1 x_2^f(\tau-\tau_{SL}^{II})},$$

$$y_2(\tau) = y_2^{SLII},$$

and

$$p_1(\tau) = p_1^{SLII} \exp\left\{-a_{12} y_2^{SLII}(\tau-\tau_{SL}^{II}) - \frac{a_{11} y_1^{SLII}}{b_1 x_2^f}\left[e^{b_1 x_2^f(\tau-\tau_{SL}^{II})} - 1\right]\right\}$$

$$\text{with } p_1^{SLII} = \frac{1}{x_1^{SLII}}\left(\frac{x_1^f}{y_1^f}\right),$$

$$p_2(\tau) = p_2^{SLII} - b_1 y_1^{SLII}\left(\frac{x_1^f}{y_1^f}\right)\left\{\frac{a_{11}}{(b_1 x_2^f)^2} + \left[q_1^{SLII}\left(\frac{y_1^f}{x_1^f}\right) + \frac{a_{11}}{b_2 x_2^f}\right](\tau-\tau_{SL}^{II}) - \frac{a_{11}}{(b_1 x_2^f)^2} e^{b_1 x_2^f(\tau-\tau_{SL}^{II})}\right\},$$

$$q_1(\tau) = q_1^{SLII} e^{-b_1 x_2^f(\tau-\tau_{SL}^{II})} - \left(\frac{x_1^f}{y_1^f}\right)\frac{a_{11}}{b_1 x_2^f}\left\{1 - e^{-b_1 x_2^f(\tau-\tau_{SL}^{II})}\right\},$$

$$q_2(\tau) = q_2^{SLII} - a_{12}\left(\frac{x_1^f}{y_1^f}\right)(\tau-\tau_{SL}^{II}).$$

Table I. (cont.) - 5

6. $P^I_{A2}: \begin{cases} u^*(\tau) = 1 \\ v^*(\tau) = 0 \end{cases}$ for $\tau_v \le \tau \le \tau^I_u$

Results are similar to those for P^I_{A1} above in 4.

7. $P^I_{A3}: \begin{cases} u^*(\tau) = 1 \\ v^*(\tau) = 0 \end{cases}$ for $\tau^*_v \le \tau \le \tau^I_u$

Results are similar to those for P^I_{A1} above in 4.

8. $P^I_{S2}: \begin{cases} u^*(\tau) = b_2/(b_1+b_2)\cdot(1-q_2y_2/(p_2x_2)) = u^*_S \\ v^*(\tau) = 0 \end{cases}$ for $\tau^*_v \le \tau \le \bar{\tau}^I_u$

As usual, we have that $S_v(\tau) < 0$ for $\tau > \tau_v$. In order for a U-singular subarc to be possible for $\tau \ge \tau^*_v$ the following condition must hold at $\tau = \tau^{*-}_v$

$$b_1p_2(\tau^{*-}_v)x_2(\tau^*_v) \ge b_2(-q_2(\tau^{*-}_v))y_2(\tau^*_v).$$

Also, on P^I_{S2} we have†

$$\frac{dx_1}{d\tau} = a_{11}x_1y_1 \qquad \text{with } x_1(\tau=\tau^*_v) = x^{v*}_1,$$

$$\frac{dx_2}{d\tau} = a_2y_2 \qquad \text{with } x_2(\tau=\tau^*_v) = x^{v*}_2,$$

$$\frac{dy_1}{d\tau} = u^*_Sb_1x_2y_1 \qquad \text{with } y_1(\tau=\tau^*_v) = y^{v*}_1,$$

$$\frac{dy_2}{d\tau} = (1-u^*_S)b_2x_2y_2 \qquad \text{with } y_2(\tau=\tau^*_v) = y^{v*}_2,$$

and

$$\frac{dp_1}{d\tau} = -a_{11}y_1p_1 \qquad \text{with } p_1(\tau=\tau^*_v) = p^{v*}_1,$$

$$\frac{dp_2}{d\tau} = -b_1y_1q_1 \qquad \text{with } p_2(\tau=\tau^*_v) = p^{v*}_2,$$

$$\frac{dq_1}{d\tau} = -a_{11}\frac{x^f_1}{y^f_1} - u^*_Sb_1x_2q_1 \qquad \text{with } q_1(\tau=\tau^*_v) = q^{v*}_1,$$

$$\frac{dq_2}{d\tau} = -a_2p_2 - (1-u^*_S)b_2x_2q_2 \qquad \text{with } q_2(\tau=\tau^*_v) = q^{v*}_2.$$

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†A further discussion of the continuity of the adjoint variables is to be found in Section 5.2.4. below.

Table I. (cont.) - 6

9. P^{II}_{B1}: $\begin{cases} u^*(\tau) = 1 \\ v^*(\tau) = 1 \end{cases}$ for $0 \le \tau \le \tau_u$ with $\frac{y_1^f}{y_2^f} < \left(\frac{y_1^f}{y_2^f}\right)^*$

τ_u is the smallest positive root of

$$\left(b_1 - \frac{a_{11}y_1^f}{x_2^f}\right) - a_{12}b_2y_2^f\tau_u + \frac{a_{11}y_1^f}{x_2^f} e^{b_1x_2^f\tau_u} = 0.$$

It should be noted that $\frac{\partial\tau_u}{\partial r} > 0$, where $r = y_1^f/y_2^f$. It may be shown that for $0 < \frac{y_1^f}{y_2^f} < \left(\frac{y_1^f}{y_2^f}\right)^*$

$$\frac{\left(\frac{b_1}{b_2}\right)}{a_{12}y_2^f} < \tau_u < \tau_u^*,$$

where the determination of τ_u^* is given in 1. We also have that

$\tau_u(r_1) < \tau_u(r_2)$ for $r_1 < r_2$ (x_1^f and x_2^f held constant). The solutions

to the state and adjoint equations are the same as those for P^{II}_{S0} given above.

Let $S_v(\tau=\tau_u) = S_v^u$, $p_1(\tau=\tau_u) = p_1^u$, etc.

10. P^{II}_{B2}: $\begin{cases} u^*(\tau) = 0 \\ v^*(\tau) = 1 \end{cases}$ for $\tau_u \leq \tau \leq \tau_v$

It follows that for all $\tau > \tau_u$ we have $S_u(\tau) < 0$ and $\frac{y_1}{y_2}(\tau) <$ $a_{12}b_2/(a_{11}b_1)$. τ_v is the smallest positive root of $S_v(\tau=\tau_v) = 0$, where $S_v(\tau)$ is given by

$$S_v(\tau) = S_v^u + a_2 b_2 y_2^u \left(\frac{x_1^f}{y_1^f}\right)\left\{\frac{a_{12}}{(b_2x_2^f)^2} + \left[q_2^u\left(\frac{y_1^f}{x_1^f}\right) + \frac{a_{12}}{b_2x_2^f}\right](\tau-\tau_u) - \frac{a_{12}}{(b_2x_2^f)^2}\exp[b_2x_2^f(\tau-\tau_u)]\right\}.$$

Also, on P^{II}_{B2} we have

$$x_1(\tau) = x_1^u \exp\{a_{11}y_1^u(\tau-\tau_u) + \frac{a_{12}y_2^f}{b_2x_2^f}[e^{b_2x_2^f(\tau-\tau_u)}-1]\},$$

$$x_2(\tau) = x_2^f,$$

$$y_1(\tau) = y_1^u,$$

$$y_2(\tau) = y_2^f \exp\{b_2x_2^f(\tau-\tau_u)\},$$

and

$$p_1(\tau) = p_1^u \exp\{-a_{11}y_1^u(\tau-\tau_u) - \frac{a_{12}y_2^f}{b_2x_2^f}[e^{b_2x_2^f(\tau-\tau_u)}-1]\} \quad \text{with} \quad p_1^u = \frac{1}{x_1^u}\left(\frac{x_1^f}{y_1^f}\right),$$

$$p_2(\tau) = p_2^u - b_2y_2^f\left(\frac{x_1^f}{y_1^f}\right)\left\{\frac{a_{12}}{(b_2x_2^f)^2} + \left[q_2^u\left(\frac{y_1^f}{x_1^f}\right) + \frac{a_{12}}{b_2x_2^f}\right](\tau-\tau_u) - \frac{a_{12}}{(b_2x_2^f)^2}e^{b_2x_2^f(\tau-\tau_u)}\right\},$$

$$q_1(\tau) = q_1^u - a_{11}\left(\frac{x_1^f}{y_1^f}\right)(\tau-\tau_u),$$

$$q_2(\tau) = \left(\frac{x_1^f}{y_1^f}\right)\left\{-\frac{a_{12}}{b_2x_2^f} + \left[\frac{a_{12}}{b_2x_2^f} + q_2^u\left(\frac{y_1^f}{x_1^f}\right)\right]e^{-b_2x_2^f(\tau-\tau_u)}\right\}.$$

Table I. (cont.) - 8

11. P_{B3}^{II}: $\begin{cases} u^*(\tau) = 0 \\ v^*(\tau) = 1 \end{cases}$ for $\tau_{SL}^{II} \leq \tau \leq \tau_v$

τ_v is the smallest positive root of $S_v(\tau=\tau_v) = 0$, where

$$S_v(\tau) = S_v^{SLII} + a_2 b_2 y_2^{SLII}\left(\frac{x_1^f}{y_1^f}\right)\left\{\frac{a_{12}}{(b_2x_2^f)^2} + \left[q_2^{SLII}\left(\frac{y_1^f}{x_1^f}\right) + \frac{a_{12}}{b_2x_2^f}\right](\tau-\tau_{SL}^{II}) - \frac{a_{12}}{(b_2x_2^f)^2}\, e^{b_2x_2^f(\tau-\tau_{SL}^{II})}\right\}.$$

Again, an upper bound on τ_v is given by $a_{12}/(a_2b_1)$. It may be shown that $S_u(\tau) < 0$ for all $\tau > \tau_{SL}^{II}$. Also, on P_{B3}^{II} we have

$$x_1(\tau) = x_1^{SLII} \exp\{a_{11}y_1^{SLII}(\tau-\tau_{SL}^{II}) + \frac{a_{12}y_2^{SLII}}{b_2x_2^f}[e^{b_2x_2^f(\tau-\tau_{SL}^{II})}-1]\},$$

$$x_2(\tau) = x_2^f,$$

$$y_1(\tau) = y_1^{SLII},$$

$$y_2(\tau) = y_2^{SLII} \exp\{b_2x_2^f(\tau-\tau_{SL}^{II})\},$$

and

$$p_1(\tau) = p_1^{SLII}\exp\{-a_{11}y_1^{SLII}(\tau-\tau_{SL}^{II}) - \frac{a_{12}y_2^{SLII}}{b_2x_2^f}[e^{b_2x_2^f(\tau-\tau_{SL}^{II})}-1]\} \text{ with } p_1^{SLII} = \frac{1}{x_1^{SLII}}\left(\frac{x_1^f}{y_1^f}\right),$$

$$p_2(\tau) = p_2^{SLII} - b_2y_2^{SLII}\left(\frac{x_1^f}{y_1^f}\right)\left\{\frac{a_{12}}{(b_2x_2^f)^2} + \left[q_2^{SLII}\left(\frac{y_1^f}{x_1^f}\right) + \frac{a_{12}}{b_2x_2^f}\right](\tau-\tau_{SL}^{II}) - \frac{a_{12}}{(b_2x_2^f)^2}\, e^{b_2x_2^f(\tau-\tau_{SL}^{II})}\right\},$$

$$q_1(\tau) = q_1^{SLII} - a_{11}\left(\frac{x_1^f}{y_1^f}\right)(\tau-\tau_{SL}^{II}),$$

$$q_2(\tau) = \left(\frac{x_1^f}{y_1^f}\right)\left\{-\frac{a_{12}}{b_2x_2^f} + \left[\frac{a_{12}}{b_2x_2^f} + q_2^{SLII}\left(\frac{y_1^f}{x_1^f}\right)\right] e^{-b_2x_2^f(\tau-\tau_{SL}^{II})}\right\}.$$

Table I. (cont.) - 9

12. $P_{B4}^{I}: \left\{ \begin{array}{l} u^*(\tau) = 0 \\ v^*(\tau) = 0 \end{array} \right.$ for $\tau_v^* \leq \tau$

It may be shown that $S_u(\tau) < 0$ and $S_v(\tau) < 0$ for all $\tau > \tau_v^*$. Also, on P_{B4}^{I} we have

$$x_1(\tau) = x_1^{v*} \exp\{a_{11} y_1^{v*} (\tau - \tau_v^*)\},$$

$$x_2(\tau) = \begin{cases} \sqrt{(x_2^{v*})^2 - \dfrac{2a_2}{b_2} y_2^{v*}} \coth(-A(\tau-\tau_v^*)+B) & \text{for } \dfrac{b_2}{2}(x_2^{v*})^2 > a_2 y_2^{v*}, \\ x_2^{v*}/\{1 - \dfrac{b_2}{2} x_2^{v*}(\tau-\tau_v^*)\} & \text{for } \dfrac{b_2}{2}(x_2^{v*})^2 = a_2 y_2^{v*}, \\ \sqrt{\dfrac{2a_2}{b_2} y_2^{v*} - (x_2^{v*})^2} \tan(C(\tau-\tau_v^*)+D) & \text{for } \dfrac{b_2}{2}(x_2^{v*})^2 < a_2 y_2^{v*}, \end{cases}$$

$$y_1(\tau) = y_1^{v*},$$

$$y_2(\tau) = \begin{cases} \left(\dfrac{b_2 (x_2^{v*})^2}{2a_2} - y_2^{v*}\right) / \sinh^2(-A(\tau-\tau_v^*)+B) & \text{for } \dfrac{b_2}{2}(x_2^{v*})^2 > a_2 y_2^{v*}, \\ y_2^{v*}/\{1 - \dfrac{b_2}{2} x_2^{v*}(\tau-\tau_v^*)\}^2 & \text{for } \dfrac{b_2}{2}(x_2^{v*})^2 = a_2 y_2^{v*}, \\ (y_2^{v*} - \dfrac{b_2}{2a_2}(x_2^{v*})\ )/\cos^2(C(\tau-\tau_v^*)+D) & \text{for } \dfrac{b_2}{2}(x_2^{v*})^2 < a_2 y_2^{v*}, \end{cases}$$

$$\text{where } A = \frac{b_2}{2}\sqrt{(x_2^{v*})^2 - \frac{2a_2}{b_2} y_2^{v*}},$$

12. P_{B4}^{I}: (concluded)

$$B = \coth^{-1}\left(\frac{x_2^{v*}}{\sqrt{(x_2^{v*})^2 - \frac{2a_2}{b_2} y_2^{v*}}}\right),$$

$$C = \frac{b_2}{2}\sqrt{\frac{2a_2}{b_2} y_2^{v*} - (x_2^{v*})^2} \ ,$$

$$D = \tan^{-1}\left(\frac{x_2^{v*}}{\sqrt{\frac{2a_2}{b_2} y_2^{v*} - (x_2^{v*})^2}}\right)$$

and

$$p_1(\tau) = p_1^{v*} \exp\{-a_{11} y_1^{v*}(\tau - \tau_v^*)\} \quad \text{with} \quad p_1^{v*} = \frac{1}{x_1^{v*}}\left(\frac{x_1^f}{y_1^f}\right) ,$$

$$\frac{dp_2}{d\tau} = -b_2 y_2 q_2 \qquad \text{with} \quad p_2(\tau = \tau_v^*) = p_2^{v*},$$

$$q_1(\tau) = q_1^{v*} - a_{11}\left(\frac{x_1^f}{y_1^f}\right)(\tau - \tau_v^*),$$

$$\frac{dq_2}{d\tau} = -a_2 p_2 - b_2 x_2 q_2 \qquad \text{with} \quad q_2(\tau = \tau_v^*) = q_2^{v*}.$$

We have not been able to develop solutions in terms of "elementary" functions to the equations for p_2 and q_2.

Table I. (concluded) - 11

13. P_{B2}^{I} and P_{B3}^{I}: $\begin{cases} u^*(\tau) = 0 \\ v^*(\tau) = 0 \end{cases}$ for $\tau_v \leq \tau$

Results are similar to those for P_{B4}^{I} above in 12.

14. P_{B5}^{I}: $\begin{cases} u^*(\tau) = 0 \\ v^*(\tau) = 0 \end{cases}$ for $\tau_{SL}^{I} \leq \tau$

Results are similar to those for P_{B4}^{I} above in 12.

15. P_{A4}^{I}: $\begin{cases} u^*(\tau) = 1 \\ v^*(\tau) = 0 \end{cases}$ for $\tau_{SL}^{I} \leq \tau \leq \tau_{u}^{I}$

Results are similar to those for P_{A1}^{I} above in 4.

5.2.3. Numerical Examples.

A computer program to calculate numerical values for information given in Table I was written in FORTRAN for the IBM 360 computer.† A plot of the field of extremals (see Figures 5, 6, and 7 below) is generated by this program. The closed-form analytic results presented in Table I are used whenever possible. Approximate numerical solutions to transcendental equations (for the determination of, for example, τ_u^*, τ_v, etc.) are developed by the well-known Newton-Raphson method. In those cases for which closed-form solutions are not available to the state and adjoint equations, a standard fourth order Runge-Kutta numerical integration method is used. A time step, $\Delta\tau$, was used in these numerical integrations which yielded agreement to the fifth place to the right of the decimal place in test cases in which the approximate numerical solution could be compared with the exact solution.

Parameter sets for the numerical examples given in this paper are shown in Table II. For our problem (2) we may consider time to be an additional state variable so that the state space is five dimensional, i.e. the state variables are t, x_1, x_2, y_1, and y_2. Thus, unfortunately, we cannot graphically depict the field of extremal trajectories but must be satisfied with viewing "cross-section" plots of it.

Table II. Parameter Sets Used to Generate Numerical Results Shown in Figures 5, 6, and 7.

Parameter Set	a_{11}	a_{12}	a_2	b_1	b_2	x_1^f	x_2^f	y_2^f
1	0.003	0.006	0.01	0.004	0.005	4.0	8.0	8.964
2	0.003	0.006	0.01	0.004	0.005	4.0	8.0	11.597

†The author would like to thank Captain Jeffrey L. Ellis (U. S. Army) for doing this work. Subsequent computational contributions were made by Captain Robert J. Hill, III (U. S. Army).

The most illuminating plot for gaining insight into the structure of the optimal fire support strategies for (2) is that of extremal trajectories in terms of y_1/y_2 versus backwards time, τ. This is shown for parameter set 1 in Figure 5. The corresponding strategic variable values for X and Y (i.e. u^* and v^*) along each extremal are also given. Other plots have been considered, but they provide little, if any, additional insight.

The most significant features of the field of extremals shown in Figure 5 are the two U-singular "surfaces": there is one in $\underset{\sim}{x},\underset{\sim}{y} - \underset{\sim}{p},\underset{\sim}{q}$ space in V-phase I and one in $\underset{\sim}{y}$-space in V-phase II. In each phase, X uses the strategy $U^* = 1$ above the singular "surface" and the strategy $U^* = 0$ below it. Similar to our discussion in [32], the singular surfaces are present in the field of optimal trajectories so that the X artillery avoids "overkilling" either Y_1 or Y_2. This insight is obvious when one, for example, considers

$$\frac{\left(-\frac{dy_1}{dt}\right)}{x_2} = b_1 y_1 .$$

Thus, the rate of destruction of Y_1 per unit of X artillery decreases over time as the Y_1 force level decreases (see [31] and [32]).

Results for parameter set 2 are shown in Figure 6. There is a void (see p. 169 and also p. 187 of [21]) in the field of extremals. This is because in backwards time at the end τ_v^* of the U-singular subarc in V-phase II, we would have $u_S(\tau_v^{*+})$ (as given in Figure 3) equal to 1.054 if the adjoint variables were continuous at τ_v^*. The following theorem further explains this situation.

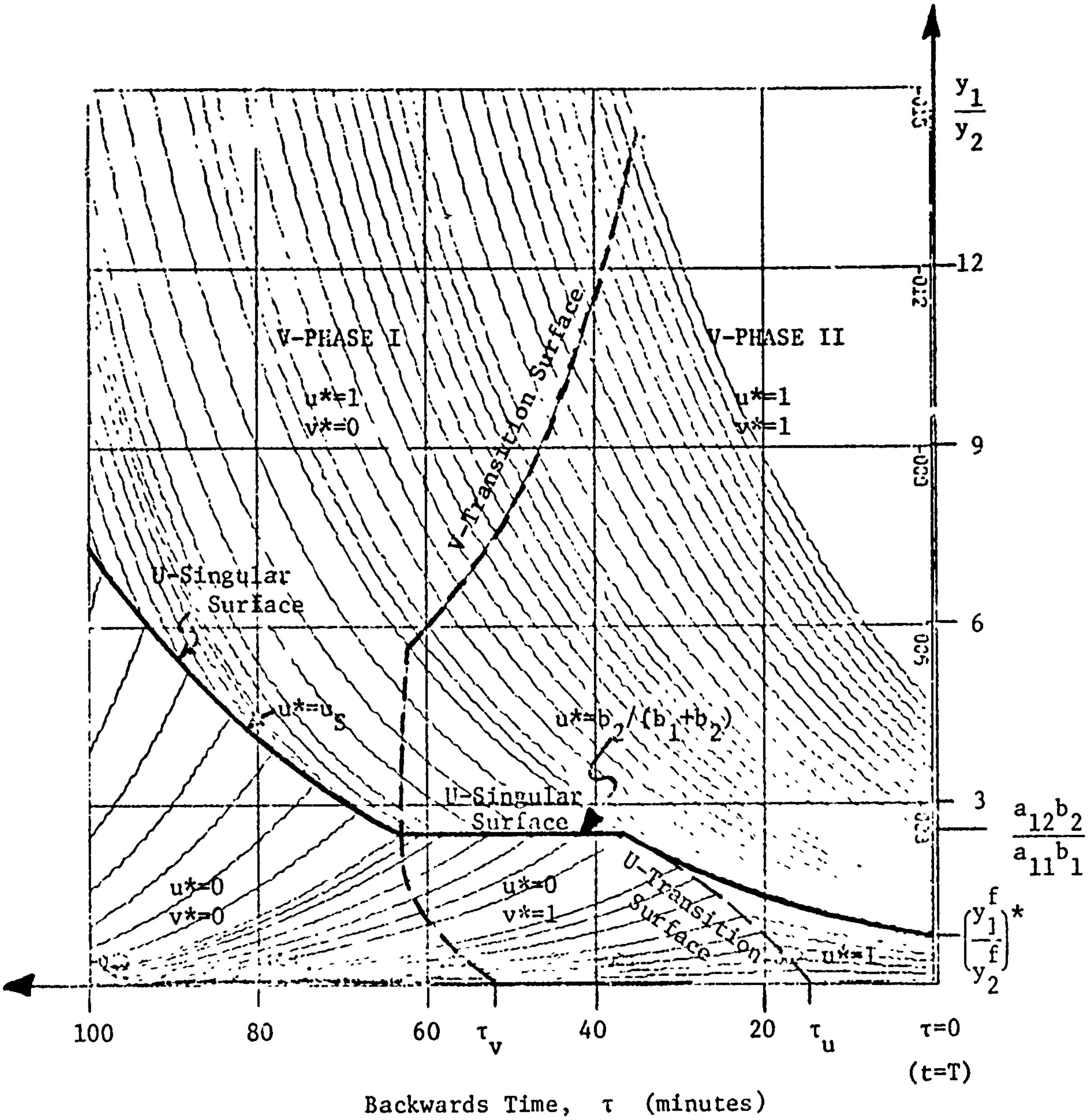

Figure 5. Plot of $\dfrac{y_1}{y_2}$ versus Backwards Time, τ, for Field of Extremals for Parameter Set 1.

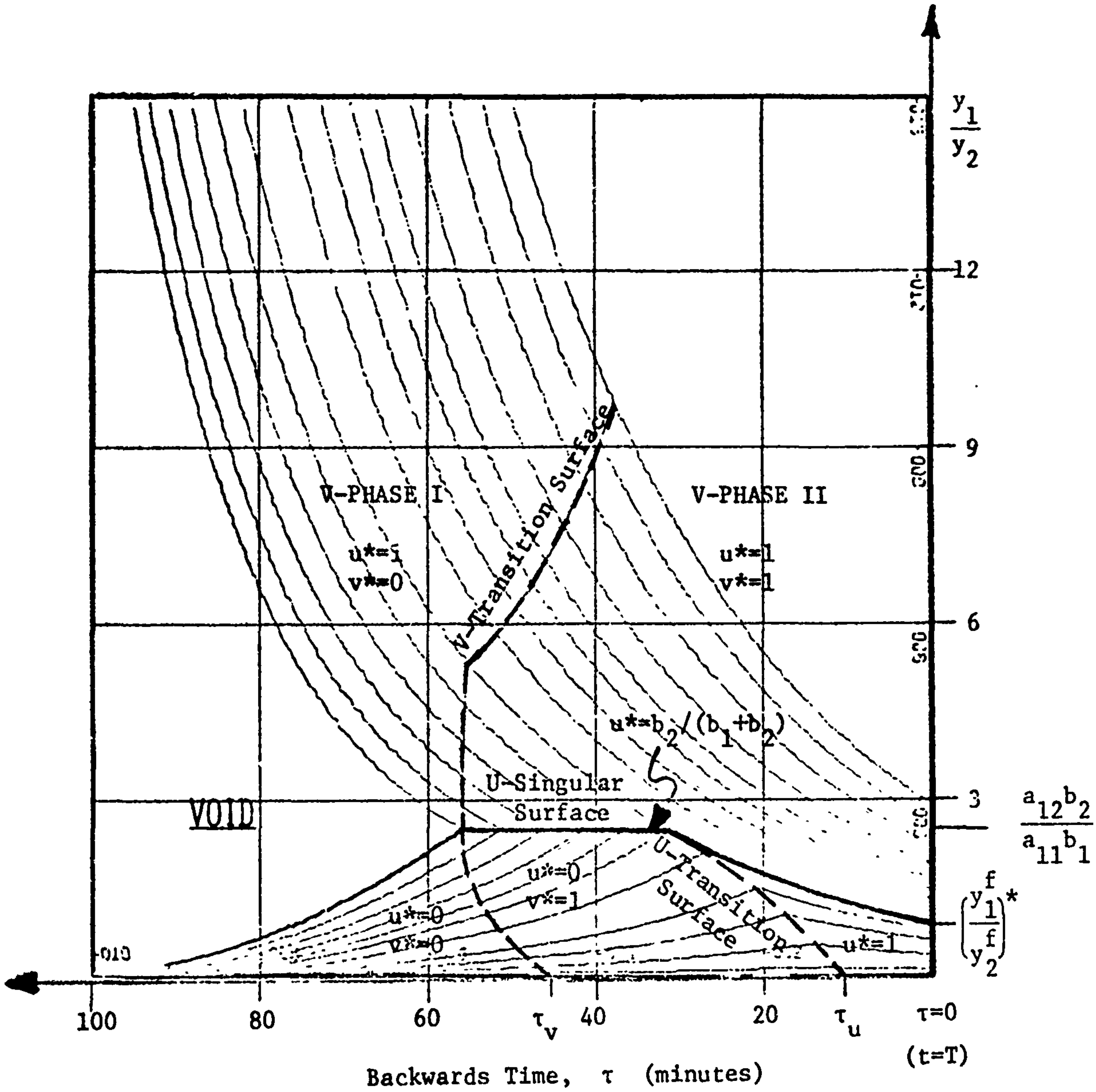

Figure 6. Void in Field of Extremals Shown in Plot of $\frac{y_1}{y_2}$ versus Backwards Time, τ, for Parameter Set 2.

THEOREM 2: There can be no U-singular subarc beginning in backwards time at τ_v^+ with $a_{11}b_1y_1 = a_{12}b_2y_2$ for

$$b_1p_2(\tau_v^-)x_2 + b_2q_2(\tau_v^-)y_2 < 0.$$

When a U-singular subarc begins at τ_v^+ with $a_{11}b_1y_1 = a_{12}b_2y_2$, there is no discontinuity in the adjoint variables at $\tau = \tau_v$ (i.e. $\sigma = 0$ in (37)).

PROOF: Immediate by (27) and (37).

Q.E.D.

Additionally, Theorem 3 gives the extremal transitions in X's strategy possible from the U-singular surface in V-phase II as we work backwards from τ_v^*. Thus, since $b_1p_2(\tau_v^*)x_2 < b_2(-q_2(\tau_v^*))y_2$ for parameter set 2, a void would exist in the field of extremals if the adjoint variables were continuous at τ_v^*.

THEOREM 3: Assume that there is no discontinuity in the adjoint variables at $\tau = \tau_v$ with $a_{11}b_1y_1 = a_{12}b_2y_2$. Then

I. if $b_1p_2(\tau_v)x_2 < b_2(-q_2(\tau_v))y_2$, then we can only have $u^*(\tau) = 0$ for $\tau \in (\tau_v, \tau_v+\delta)$ where $\delta > 0$,

II. if $b_1p_2(\tau_v)x_2 \geq b_2(-q_2(\tau_v))y_2$, then we can have

$$u^*(\tau) = \begin{cases} \text{(a)} & 0, \\ \text{(b)} & (1-q_2y_2/(p_2x_2)) \cdot b_2/(b_1+b_2), \\ \text{(c)} & 1, \end{cases}$$

for $\tau \in (\tau_v, \tau_v+\delta)$ where $\delta > 0$.

PROOF: (a) When we are on the singular surface in V-phase II at $\tau_v^- = \tau_v^{*-}$, then by (22) and (23) and the continuity of the dual variables we have

$$S_u(\tau=\tau_v^+) = \overset{\circ}{S}_u(\tau=\tau_v^+) = 0,$$

and

$$\ddot{S}_u(\tau=\tau_v^+) = a_2 b_2 (b_1+b_2) p_2 x_2 y_2 \{u^*(\tau_v^+) - \left(\frac{b_2}{b_1+b_2}\right)\left(1 - \frac{q_2 y_2}{p_2 x_2}\right)\} , \tag{47}$$

where $\dot{S}_u$ denotes $\frac{dS_u}{d\tau}$.

(b) Considering a Taylor series expansion about $\tau = \tau_v^+$, we have by the above for $\tau \geq \tau_v^+ = \tau_v^{*+}$

$$S_u(\tau) = \frac{(\tau-\tau_v^*)^2}{2} \ddot{S}_u(\bar{\tau}), \tag{48}$$

where $\bar{\tau} \in (\tau_v^*, \tau)$.

(c) When $u^*(\tau) = 0$ for $\tau \in (\tau_v^*, \tau_v^*+\delta)$, then

$$\ddot{S}_u(\tau=\tau_v^{*+}) = -a_2 (b_2)^2 p_2 x_2 y_2 \left(1 - \frac{q_2 y_2}{p_2 x_2}\right) < 0,$$

so that $\exists\ \delta_1 > 0$ such that $S_u(\tau) < 0$ for all $\tau \in (\tau_v^*, \tau_v^*+\delta_1)$. Thus, we can always have $u^* = 0$ as we work backwards in V-phase I from the U-singular subarc in V-phase II.

(d) Now let $b_1 p_2(\tau_v^*) x_2 \geq b_2(-q_2(\tau_v^*)) y_2$. By (26), the U-singular control in V-phase I $u_S = (1-q_2 y_2/(p_2 x_2)) \cdot b_2/(b_1+b_2) \leq 1$. Thus, the U-singular subarc is possible. When $u^*(\tau_v^{*+}) = 1$, then $\ddot{S}_u(\tau=\tau_v^{*+}) \geq 0$ by (47). When inequality holds, it follows that $\exists\ \delta_1 > 0$ such that $S_u(\tau) > 0$ for all $\tau \in (\tau_v^*, \tau_v^*+\delta_1)$. Clearly, we cannot have $u^* = 1$ if $b_1 p_2(\tau_v^*) x_2 < b_2(-q_2(\tau_v^*)) y_2$. Q.E.D.

The same analysis as used in the proof of Theorem 3 applies on a U-singular subarc in V-phase I when $v^* = 0$. As long as (27) holds, one has three options similar to those of part II of Theorem 3.

5.2.4. Filling in a Void.

We have emphasized that H, $\underset{\sim}{p}(t)$, and $\underset{\sim}{q}(t)$ are continuous functions of time except possibly at manifolds of discontinuity of both U^* and V^* (see Section 4.3 above). From Theorem 3 it follows that a void must exist in the field of extremals when these functions are continuous and $b_1 p_2(\overset{*}{\tau_v})x_2 < b_2(-q_2(\overset{*}{\tau_v}))y_2$. At $\overset{*}{\tau_v}$, moreover, v^* changes (as we progress backwards in time) from 1 to 0 and u^* from $b_2/(b_1+b_2)$ to a different value. Thus, we have a manifold of discontinuity of both U^* and V^*. Moreover, considering results given above, it is readily shown that $u^*(\tau)$ remains for increasing τ (i.e. backwards time) equal to zero once it changes to zero. Then from Theorems 2 and 3 it follows that for $b_1 p_2(\tau_v^{*-})x_2 < b_2(-q_2(\tau_v^{*-}))y_2$ the dual variables must be discontinuous to fill in the void, and we must have $u^*(\tau) = 1$ for $\tau_v^* < \tau < \tau_u^{I*}$. Furthermore, considering Figure 6 and considerations "in the large," the manifold of discontinuity must lie on the V-transition surface.

Thus, we have established that for $a_{11}b_1y_1 = a_{12}b_2y_2$ we have

$$\begin{cases} u^*(\tau_v^-) = b_2/(b_1+b_2), \\ v^*(\tau_v^-) = 1 \end{cases} \quad \text{and} \quad \begin{cases} u^*(\tau_v^+) = 1, \\ v^*(\tau_v^+) = 0. \end{cases} \qquad (49)$$

It remains to determine the function $\tau_v(\underset{\sim}{x},\underset{\sim}{y})$ of (29) so that $\frac{\partial\tau_v}{\partial\underset{\sim}{x}}$ and $\frac{\partial\tau_v}{\partial\underset{\sim}{y}}$ may be computed, and the jumps in H, $\underset{\sim}{p}$, and $\underset{\sim}{q}$ subsequently determined (see (30) through (33)). It should be clear that it is impossible to explicitly determine $\tau_v(\underset{\sim}{x},\underset{\sim}{y})$. However, by computation of five points on the V-transition surface, the desired partial derivatives may be estimated by using linear approximations to the appropriate directional derivatives and solving a system of four linear equations in four unknowns. For parameter set 2 (as the reference case), this yielded the following estimates

$$\frac{\partial\tau_v}{\partial x_1} = 0.0000, \qquad \frac{\partial\tau_v}{\partial x_2} = -0.295,$$

$$\frac{\partial\tau_v}{\partial y_1} = -0.0167, \qquad \frac{\partial\tau_v}{\partial y_2} = -0.0331. \tag{50}$$

It is, therefore, convenient to re-write the jump conditions across the manifold of discontinuity of both U* and V*.

$$p_1(\tau_v^+) = p_1(\tau_v^-), \qquad p_2(\tau_v^+) = p_2(\tau_v^-) - \rho\,\frac{\partial\tau_v}{\partial x_2},$$

$$q_1(\tau_v^+) = q_1(\tau_v^-) - \rho\,\frac{\partial\tau_v}{\partial y_1} - \sigma a_{11}b_1, \tag{51}$$

$$q_2(\tau_v^+) = q_2(\tau_v^-) - \rho\,\frac{\partial\tau_v}{\partial y_2} + \sigma a_{12}b_2,$$

where ρ and σ are related by (36). In this case the jumps (37) and (38) in the switching functions simplify to

$$S_u(\tau_v^+) = \sigma\left\{a_{11}(b_1)^2y_1 + a_{12}(b_2)^2y_2 + \frac{a_{11}(b_1)^2y_1x_2\left(b_1y_1\,\frac{\partial\tau_v}{\partial y_1} - b_2y_2\,\frac{\partial\tau_v}{\partial y_2}\right)}{\left(1-a_2y_2\,\frac{\partial\tau_v}{\partial x_2} - b_1y_1x_2\,\frac{\partial\tau_v}{\partial y_1}\right)}\right\}, \tag{52}$$

and

$$S_v(\tau_v^+) = \frac{a_{11}a_2(b_1)^2y_1x_2\,\frac{\partial\tau_v}{\partial x_2}\,\sigma}{\left(1-a_2y_2\,\frac{\partial\tau_v}{\partial x_2} - b_1y_1x_2\,\frac{\partial\tau_v}{\partial y_1}\right)}. \tag{53}$$

Since $v^*(\tau_v^{*+}) = 0$, we must have $S_v(\tau_v^{*+}) \leq 0$ so that (50) and (53) yield that $\sigma \geq 0$. It should be clear that $\sigma = 0$ if and only if $\underset{\sim}{H}$, $\underset{\sim}{p}$, and

$\mathfrak{g}$ are continuous at τ_v^*. For $\sigma > 0$, the condition that $u^*(\tau_v^{*+}) = 1$ yields that we must have

$$\frac{S_u(\tau_v^{*+})}{\sigma} > 0, \tag{54}$$

where $S_u(\tau_v^{*+})$ is given by (52). Although it cannot in general be guaranteed that (54) will always hold when a void in the field of extremals such as that shown in Figure 6 exists, it should be clear that it must if the problem (2) is to have a solution. The author conjectures that this is true. It is readily shown that when (54) holds, we have

$$S_u(\tau_v^{*+}) > 0, \qquad \overset{\circ}{S}_u(\tau_v^{*+}) < 0, \qquad \text{and} \qquad S_v(\tau_v^{*+}) < 0. \tag{55}$$

The appropriate value for σ is determined by "considerations in the large:" the structure of the entire field of extremals determines the value of this parameter. In Figure 7, we let τ_u^{I*} denote the backwards time at which the U-singular subarc is entered in V-Phase I. Corresponding to τ_u^{I*} is σ^*, which yields the first and second conditions (18) and (25) (with $u_S^* \leq 1$) for a U-singular subarc with $V^* = 0$ at $\tau_u^{I*} > \tau_v^*$. For $0 < \sigma < \sigma^*$, one uses $u^*(\tau) = 1$ for $\tau_v^{*+} < \tau < \tau_u^I$ and then $u^*(\tau) = 0$ for $\tau > \tau_u^I$. For $\sigma > \sigma^*$, the U-switching function $S_u(\tau)$ never changes sign so that $u^*(\tau) = 1$ for all $\tau > \tau_v^*$. Thus, by manipulation of σ, one may fill in the void in the field of extremals in V-Phase I. The resulting field of extremals is shown in Figure 7.

5.2.5. <u>The Case of Negligible Y_1 Small Arms Effectiveness.</u>

It seems appropriate to consider what happens to the solution to the problem at hand as the (relative) effectiveness of Y_1 (small arms) fire becomes negligible, i.e. as $a_{11} \to 0$. Let us consider (either) Figure 5 (or Figure 7). The U-singular "surface" in V-Phase II has equation $y_1/y_2 = a_{12}b_2/(a_{11}b_1)$.

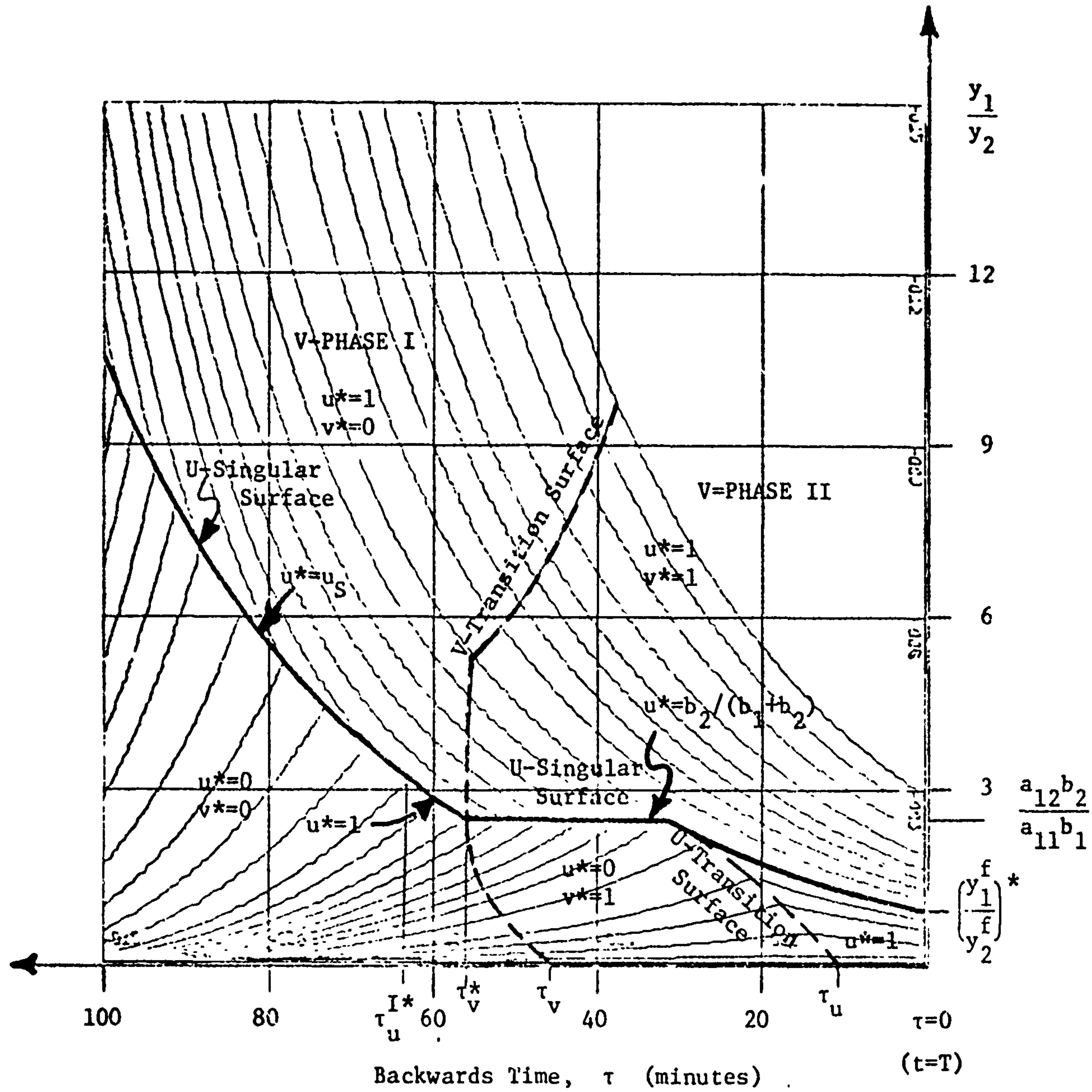

Figure 7. Filled-In Void in Field of Extremals For Parameter Set 2.

Thus, as $a_{11} \to 0$ with the other parameters being held constant, this singular "surface" appears higher and higher on the y_1/y_2 axis in Figure 5. In the limit, the singular surface does not appear in the finite part of the plane. Thus, we have shown that an optimal strategy in which a side divides the fire of its supporting weapon system between the enemy's primary (infantry) and supporting systems can only occur when the enemy's infantry has some fire effectiveness (in the sense of a non-zero Lanchester attrition-rate coefficient) against his infantry.

6. Discussion.

In this paper we have examined the dependence of optimal time-sequential fire-support strategies on the form of the combat attrition model by considering a differential game (see equations (2)) with slightly different combat dynamics than those in the fire-support differential game considered by Kawara [22] (see equations (1)). For this fire-support differential game (2) we developed first order necessary conditions of optimality and constructed "cross-section" pictures of the field of extremals. By comparing and contrasting the structure of optimal fire-support strategies for our problem (2) with that for Kawara's fire-support differential game (1), one begins to understand the nature of the dependence of optimal strategies on the combat dynamics by also comparing and contrasting the combat attrition equations for these two differential games.

Our fire-support differential game (2) was similar to Kawara's problem (1) (see [22]) except that we let the attacker's (i.e. X's) artillery produce "linear-law" attrition† against both the defender's artillery and also his infantry and let the defender's infantry produce "linear-law" attrition against the attacker's infantry. As contrasted with the optimal time-sequential fire-support

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†For convenience we use the term "linear-law" attrition to denote an attrition process in which a target-type undergoes attrition at a rate proportional to the product of the numbers of firers and targets (see [31], [32]).

strategies for Kawara's problem (1) of always concentrating all artillery fire on first enemy artillery and then later enemy infantry (the timing of the switch being force-level independent), for our problem (2) the optimal strategy for one combatant (the attacker, X) depends directly on the enemy's force levels and is no longer to always concentrate all fire on either the enemy's primary or secondary weapon system. The latter result, moreover, was shown to depend on the defender's infantry having some fire effectiveness (in the sense of a non-zero Lanchester attrition-rate coefficient) against the attacker's infantry.

The solution to (2) is characterized by the presence of singular surfaces (in Issacs' terminology (see [21]), universal surfaces (see also [18])), a different one for each V-phase of battle. When the battle state reaches one of these surfaces, X follows an optimal strategy of dividing his artillery fire between enemy infantry and artillery in order to avoid "overkill." Another characteristic of the optimal fire-support strategies (not present for Kawara's [22] problem (1)) is that X's optimal strategy may sometimes depend on Y's distribution of supporting fires. This behavior occurs on the singular surfaces. In fact, X sometimes must react instantaneously to changes in Y's fire distribution.

The development of even a partial solution to (2) has involved a solution phenomenon not previously reported for Lanchester-type differential games: the adjoint (or dual) variables† are discontinuous across a manifold of discontinuity of both U^* and V^*. This manifold of discontinuity exists for a certain range of parameter values in the solution to the problem at hand (2). Furthermore, there is a military interpretation to this manifold of discontinuity: if Y_2 concentrates fire on X_2 and X_2 on Y_1, then when Y_2 changes to concentrating all fire on X_1, X must re-evaluate the worth of a Y_2 unit because it now has

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†The reader should recall that these represent the marginal values of force types, i.e. $p_2(t) = \frac{\partial V}{\partial x_2(t)}$ where $V = V(t,\underset{\sim}{x},\underset{\sim}{y})$ denotes the value of the differential game (see [14], [21]).

a direct influence on the payoff. Such a discontinuity in the adjoint variables is unique to differential games (see [3], [4]) (i.e. it cannot occur for a one-sided optimal control problem).

It should also be pointed out that the presence of singular (i.e. universal) surfaces in the solution to (2) is apparently independent of the form of the criterion functional (here terminal payoff) and depends only on the combat dynamics. For purposes of comparison we considered the same payoff as considered by Kawara [22]. We also showed that the singular (i.e. universal) surfaces can only be present in the solution when the defender's infantry Y_1 has a nonzero casualty producing capability against X_1.

The problem (2) considered in this paper has certain similarities to the "War of Attrition and Attack: Second Version" studied by R. Isaacs (see pp. 330-335 of [21]). We have, however, developed a much more complete solution to our problem than that given in [21] for Mengel's problem. Although this problem (2) possesses some similarities to the Lanchester-type optimal control problem studied by us in [31], its solution has turned out to be much more complex. Our developments in this paper, however, have been significantly helped by intuition gained in the study of the simpler, one-sided problem (see [32] for a further discussion).

As a result of our investigation here, we hope that a better understanding of optimal fire-support strategies has been developed. As is always the case, however, the insights gained into the optimization of combat dynamics from our study of the differential game (2) are no more valid than the combat model itself. Our work here shows that the functional forms of the various target-type casualty rates produced by the artillery essentially determines the most significant aspects of the structure of the optimal fire-support strategies. Thus, our study of this optimization problem shows the importance of determining the appropriate (Lanchester-type) model of combat dynamics.

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